Name: ____________________

MATH 250 Student Number: ____________________

Spring 2022 Instructor: ____________________

Section: ____________________

Sample Final

This exam has 11 questions for a total of 150 points.

You must show work for full credit.

Only Real Valued solution are accepted.

Notes, devices, communication with other people is NOT permitted (except to ask an instructor a question).

1. (10 points) For a), b), c), d) and e) write "True" if the statement is true, or "False" if the statement is false. You must write **the full word True or False** to receive credit.

a) The ODE $y' = y(9-t)$ is both linear and separable __________

b) The ODE $y'' - 2e^t y' y = \sin(3t)$ is linear __________

c) The ODE $y'(e^t - 1) = (y^2 - 1)$ is separable __________

d) The ODE $y' = 7$ is both linear and autonomous __________

e) The ODE $\dfrac{3y' + e^t y}{\cos t} = 8y$ is both linear and separable __________

2. (10 points) Find all equilibrium / constant solutions (if any) to each ODE

a) $y'' - \dfrac{e^{ty'}}{y} = \sin(yy') + 2\cos(yy') + y$

b) $\dfrac{1}{\omega}\dfrac{d^2\omega}{d\psi^2} = 2\psi^2 e^{\psi + \omega'}$

3. (10 points) Consider the autonomous ODE: $y' = (1 - y)(3 - y)^2$

a) Find all critical points (equilibirum solutions / constant solutions) and classify their stability.

b) Suppose $y_1(t)$ is a solution satisfying the initial condition $y_1(1) = 2$.
Suppose $y_2(t)$ is a solution satisfying the initial condition $y_2(2) = 4$.
Graph the 2 solutions y_1 and y_2 and **clearly label which graph is y_1 and which is y_2.**

4. (15 points) Find the general solution to (assume $t > 0$)

$$y' - \frac{y}{t} = te^{\sqrt{t}}$$

5. (15 points) Consider the ODE

$$-\frac{1}{y^2}\,\frac{dy}{dt} = t\sqrt{t+3}$$

Find the solution to the ODE which satisfies the initial condition: $y(-3) = e^{-1}$

6. (15 points) Solve the IVP:

$$y'' - 3y' - 4y = \delta(t-1) - \delta(t-2) \qquad y(0) = 0,\, y'(0) = 8$$

7. (15 points) The position (measured in meters) of a mass tied to a spring hanging from the ceiling is modeled by the IVP:

$$u'' + 2u' + 8u = 0 \qquad u(0) = 0, u'(0) = 2$$

in which positive values of u represent positions below the equilibrium position ($u = 0$). Assume the mass begins moving at time $t = 0$.

a) Find the position of the mass $u(t)$ for all time $t \geq 0$

b) At what time is the mass at its lowest point? (Note that the mass will hit its lowest point when u attains its most positive value.)

c) How much time passes between successive passes through the equilibrium position?

8. (15 points) Solve the linear homogeneous system , $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with given initial condition, where

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ -5 & -9 \end{bmatrix}$$

and

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

9. (15 points) Find the general solution of the linear homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is defined below, classify the stability of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and fully sketch the phase plane:

for nodes/saddles, sketch all straight trajectories, plus one curved trajectory in each "quadrant", and for spirals/ellipses, neither eccentricity nor orientation is required.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

10. (15 points) Find the general solution of the linear homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is defined below, classify the stability of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and fully sketch the phase plane:

for nodes/saddles, sketch all straight trajectories, plus one curved trajectory in each "quadrant", and for spirals/ellipses, neither eccentricity nor orientation is required.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

11. (15 points) Consider the non-linear system of ODE's below:

$$\mathbf{x}' = \begin{bmatrix} y^2 e^x - e^x \\ x^3 + xy^2 \end{bmatrix}$$

a) This system has 2 constant solutions (critical points or equilibrium solutions), what are they?

b) Determine the linearization near each critical point, and classify the stability of each critical point.

TABLE 6.2.1 **Elementary Laplace Transforms**

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n$ a positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 24
5. $\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 5
7. $\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
8. $\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 6
9. $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 10
10. $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 11
11. $t^n e^{at}, \quad n$ a positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 14
12. $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 17
16. $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 21