Math 250, Section 7, Quiz 2

Name:______________________________

ID Number: ______________________________

Instructions: Clearly answer each of the questions below. Remember to check the back side. Use full sentences and proper grammer for verbal answers. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. (40 pts) Consider the following linear second-order homogeneous ordinary differential equation

$$y'' - 20y = y'.$$

(a) Find the general solution.

Answer: First, we re-arrange the equation so it is in standard form.

$$y'' - y' - 20y = 0.$$

Now, taking the ansatz $y = e^{\lambda t}$, we find the characteristic equation is $(\lambda^2 - \lambda - 20)e^{\lambda t} = 0$. Then using the quadratic formula,

$$\lambda \in \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-20)}}{2} = \frac{1 \pm \sqrt{81}}{2} = \{5, -4\}.$$

We've found two distinct real roots, so the general solution is

$$y(x) = C_1 e^{5x} + C_2 e^{-4x}$$

(b) If $y(0) = 0$ and $y'(0) = 18$, find the specific solution of this equation.

Answer:

$$y(0) = 0 = C_1 e^{5(0)} + C_2 e^{-4(0)} = C_1 + C_2$$
$$y'(0) = 18 = 5C_1 e^{5(0)} - 4C_2 e^{-4(0)} = 5C_1 - 4C_2$$

Since $0 = C_1 + C_2$, then $C_2 = -C_1$. Substituting for C_2, $18 = 5C_1 - 4(-C_1) = 5C_1 + 4C_1 = 9C_1$. Thus, $C_1 = 18/9 = 2$ and $C_2 = -C_1 = -2$. Thus, we the specific solution matching the given initial conditions is

$$y(x) = 2e^{5x} - 2e^{-4x}.$$

2. (20 pts) Classify the following equation, and determine the largest interval over which a solution through the point $y(3) = 2$ can be uniquely constructed.

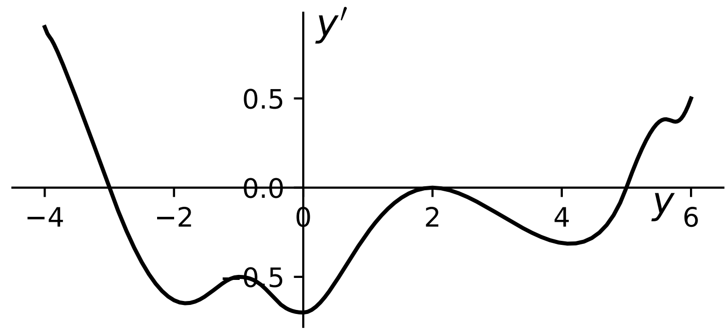
$$(\cos x)y' + (x^2 - 9)y = e^{19x}$$

Answer: This is a 1st-order linear ordinary differential equation, non-autonomous, not separable. In standard form,

$$y' + \frac{x^2 - 9}{\cos x} y = \frac{e^{19x}}{\cos x}$$

There are singularities when $\cos(x) = 0$. These happen when $x = n\pi + \pi/2$ for $n \in \mathbb{Z}$, so the largest interval containing $x = 3$ where we can construct the solution is $(\pi/2, 3\pi/2) \approx (1.5, 4.5)$.

3. (20 pts) Suppose we have an autonomous ordinary differential equation where the equation $y' = f(y)$, with f given in the figure. Find all steady-state solutions, and classify each as stable, unstable, or semistable.



Answer: There are three steady-state solutions where $y' = 0$. The steady-state $y(t) = -3$ is stable (because things a little smaller increase towards -3, while things a little larger decrease towards -3), the steady-state $y(t) = 2$ is semistable, and the steady-state $y(t) = 5$ is unstable (because initial conditions slightly smaller continue to decrease away from it, and initial conditions slightly larger than 5 increase away from it.

4. (20 pts) What special properties (discussed in Section 3.2) do linear equations have that make them easier to solve than nonlinear equations?

Answer: Linear equations are generally **NOT SEPARABLE**. Linear equations are easier to solve than nonlinear equations, but they are actually quite rare in real-world modelling. Characteristic equations are only helpful when the linear equation is constant-coefficient; equations like $y'' - xy = 0$ can not be solved using characteristic equations.

The solutions of linear equations can be constructed by adding different solutions together, thanks to the super-position principle. The general superposition principle says that for any linear operator L, if $L[x] = a$ and $L[y] = b$, then $L[x + y] = a + b$. Specifically, for linear 2nd order homogenous ordinary differential equations, when

$$y_1'' + p(x)y_1' + q(x)y_1 = 0,$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

then $y(x) = C_1 y_1(x) + C_2 y_2(x)$ also solves

$$y'' + p(x)y' + q(x)y = 0$$