Math 250, Section 7, Quiz 3

Name:____

ID Number: ____

Instructions: Clearly answer each of the questions below. Remember to check the back side. Use full sentences and proper grammer for verbal answers. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. Consider the differential equation

$$y'' - 64y = 320x^2 + 246, \quad y(0) = -17, \quad y'(0) = 200.$$

(a) Find a particular solution using the method of undetermined coefficients.

Answer:

$$-64y(x) + \frac{d^2}{dx^2}y(x) = 320x^2 + 246$$
$$64C_0 - 64C_1x + 64C_2x^2 - 320x^2 + \frac{\partial^2}{\partial x^2}\left(-C_0 + C_1x - C_2x^2\right) - 246$$
$$2 \cdot \left(32C_0 - 32C_1x + 32C_2x^2 - C_2 - 160x^2 - 123\right)$$
$$64C_0 - 64C_1x - 2C_2 + x^2 \cdot (64C_2 - 320) - 246$$
$$[64C_0 - 2C_2 - 246,\ -64C_1,\ 64(C_2 - 5)]$$
$$\{C_0 : 4,\ C_1 : 0,\ C_2 : 5\}$$
$$y_p(x) = -5x^2 - 4$$

(b) Find the general solution of the corresponding homogenized equation.

Answer:

$$y_h(x) = C_1e^{8x} + C_2e^{-8x}$$

(c) Find the specific solution to the given problem.

Answer:

$$y(x) = C_0e^{8x} + C_1e^{-8x} - 5x^2 - 4$$
$$C_0 + C_1 + 13 = 0$$
$$8C_0 - 8C_1 - 200 = 0$$
$$\{C_0 : 6,\ C_1 : -19\}$$
$$y(x) = 6e^{8x} - 19e^{-8x} - 5x^2 - 4$$

2. A spring with relaxed length 1 meter hangs from the ceiling of a hanger on a planet where gravity's acceleration is $10m/s^2$.

   (a) A weight of mass 3 kg is hung from the spring. The spring stretches so that it's new rest length is 39/34 meters. What is the spring constant?

   Answer: In the basic model, the position $y(t)$ for a weight hanging from a spring satisfies $m\ddot{y} = -k(y - \ell) - \gamma\dot{y} - mg$. Here, $g = 10$ meters per square second, $\ell = 1$ meter and $m = 3$ kilograms, and when at rest, $y = 39/34$, $\dot{y} = 0$, $\ddot{y} = 0$. From basic model equation, then, so we must have $0 = -k(y - 1) + gm$, so

   $$k = (mg)/(y - 1) = 10 \times 3/((39/34) - 1) = 30 \times (34/5) = 204$$

   (b) On this planet, if the drag coefficient $\gamma = 12$, what would be the free-fall speed of our weight?

   Answer: $F_g = F_d$, $mg = \gamma\dot{y}$, $10 \times 3 = 12\dot{y}$, $\dot{y} = 5/2$

   (c) Formulate the initial value problem for the **displacement of the weight from rest position** when the weight starts with initial displacement of 2 m from rest and with an initial velocity of -36 m/s.

   Answer: The displacement equation is $m\ddot{u} + \gamma\dot{y} + ky = 0$. Using our parameter values,

   $$3u'' + 12u' + 204u = 0, \quad u(0) = 2, \; u'(0) = -36.$$

   (d) Show that this spring system is under-damped and determine the quasi-period of oscillations?

   Answer: Solving the characteristic equation $3\lambda^2 + 12\lambda + 204 = 0$, we find $\lambda = -2 \pm 8i$. Since the roots are complex numbers, the system is underdamped. So, the quasi-frequency $\omega_q = 8$. The quasiperiod $T_q = \frac{2\pi}{\omega} = \pi/4$.

   (e) Find the general solution for the displacement of the weight.

   Answer:

   $$u(t) = e^{-2t}\left(C_1 \sin(8t) + C_2 \cos(8t)\right)$$