Name:____________________________________ Math 250, Section 7, Quiz 6

Instructions: Clearly answer each of the questions below. Remember to check the back side. Use full sentences and proper grammer for verbal answers. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. Consider the linear system

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -9 & -1 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(a) (1 pt) What kind of steady-state (sink, source, or saddle) is the steady-state $(0,0)$?

The characteristic equation is $(\lambda+9)^2+4=0$. The eigenvalues are $\lambda=-9\pm 2i$, both of which have negative real part, meaning the origin $y(t)=[0,0]^T$ is stable. Alternatively, we could observe that the trace is -18, and that the determinate is 85, which imply stability as well.

(b) (5 pts) Find the general solution in terms of real numbers.

Let's pick an eigenvalue. Let $\lambda=-9+2i$. The eigenvector v must solve $(A-\lambda I)v=0$, so

$$(A-\lambda I)v = \left(\begin{bmatrix} -9 & -1 \\ 4 & -9 \end{bmatrix} - (-9+2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then $(A-\lambda I)v=0$ implies

$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C\begin{bmatrix} 1 \\ -2i \end{bmatrix}.$$

So any vector of the form $[v_1, v_2] = C[1,-2i]^T$ will be an eigenvector. Thus, one space of solutions, in complex-number form, is

$$y(t) = Ce^{(-9+2i)t}\begin{bmatrix} 1 \\ -2i \end{bmatrix}.$$

How, we showed in class that the real if $y=u(t)+iv(t)$ is a solution, then so is $C_u u(t)+C_v v(t)$.

$$e^{-9t}e^{2it}\begin{bmatrix} 1 \\ -2i \end{bmatrix} = e^{-9t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1 \\ -2i \end{bmatrix} = e^{-9t}\begin{bmatrix} \cos 2t + i\sin 2t \\ 2\sin 2t - 2i\cos 2t \end{bmatrix}$$

Thus, in terms of real-valued functions, the following is also a solution.

$$C_1 e^{-9t}\begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix} + C_2 e^{-9t}\begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix}$$

Now, this is first-order system of 2 equations in 2 variables, so we expect 2 integration constants total. Thus, this must be the general solution.

2. Consider

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ -8 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

given that $\lambda = 7$ is the only eigenvalue and $[1, -2]^T$ is the only eigenvector.

(a) (1 pt) What kind of steady-state (stable or unstable) is the point solution $(0, 0)$?

The origin $(0, 0)$ must be an unstable node when $\lambda = 7 > 0$ is the only eigenvalue.

(b) (4 pts) Find the general solution.

Knowing $\lambda = 7$ is the only eigenvalue, it must be a repeated eigenvalue. For a repeated eigenvalue with only one eigenvector, we expect the general solution to have the form

$$y(t) = e^{\lambda t}\left(C_1 v + C_2(vt + w)\right)$$

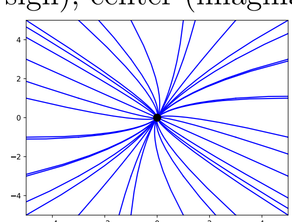
where $(A - \lambda I)v = 0$ and $(A - \lambda I)w = v$. We're told $v = [1, -2]^T$, so if $(A - \lambda I)w = v$, then

$$\begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

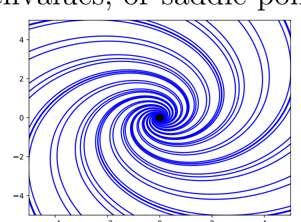
$4w_1 + 2w_2 = 1$. $w = [0, 1/2]^T$ is one possible solution, in which case

$$y(t) = C_1 e^{7t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{7t}\begin{bmatrix} t \\ -2t + 1/2 \end{bmatrix}.$$

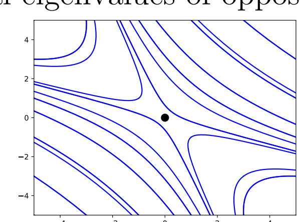
3. (3 pts) Label each of the following as a focus (complex eigenvalues), node (real eigenvalues of same sign), center (imaginary eigenvalues, or saddle point (real eigenvalues of opposite sign).



node



focus



saddle