Math 421 Homework assignment 1

due Wednesday, August 31st

1. Priestly, Chapter 1, Exercises 1.1-1.2

3. Calculate the following.

   (a) $(2I + 3i) + (-4I + i)$

   (b) $(4I + 5i) - (2I + 3i)$

   (c) $(2 + 5i)(2 - 3i)$ (We'll drop the I from now on...)

   (d) $i(4 + i)$

   (e) $(2 + 3i)(2 - 3i)$

   (f) $(2 - 3i)(2 + 3i)$

   (g) $(1 + i)^n$ for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

   (h) $(2 + 5i)^{-1}$

   (i) $\dfrac{2 + 3i}{2 - 3i}$

   (j) $(x + iy) + \overline{(x + iy)}$, where $x, y \in \mathbb{R}$.

   (k) $\overline{(2 - i)^2}$

   (l) $|8 - 9i|$

4. The set of Gaussian integers is the subset of complex numbers where both the real and imaginary parts are integers. Find by trial and error three Gaussian integer solutions of the depressed cubic $10 - z - z^3 = 0$.

   Answer: If we graph $10 - z = z^3$, we can see that there must be exactly one real solution between $z = 0$ and $z = 4$. By trial and error, we find $z = 2$ is a solution. Now, we need complex solutions. The simplest thing is to try a bunch of points around $z = 0$. It's not to long before we find solutions $z \in -1 \pm 2i$.

5. Recall that a permutation matrix is any matrix with exactly one 1 in each row and columns, and all other entries being zero. Suppose p is a 3×3 permutation matrix with no 1's on the main diagonal, we can construct a ring N over $\mathbb{R}$ as $N = \{a + bp + cp^2 : (a, b, c) \in \mathbb{R}^3\}$. Note, here, we are using the same convention in class, where $a + bp + cp^2 = ap^0 + bp^1 + cp^2$, and $p^0 = I$, the identity matrix. (edited 2022-8-29)

   (a) Show that for the permutation p you picked, p^3 is equal to the identity matrix.
   Answer: If
   $$p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$
   then
   $$p^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
   and
   $$p^3 = (p)(p^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

(b) What is the expanded formula for $(a + bp + cp^2)(d + ep + fp^2)$?
Answer:

$$(a + bp + cp^2)(d + ep + fp^2) = (ad + bf + ce) + p\,(ae + bd + cf) + p^2\,(af + be + cd)$$

(c) What is the formula for $(a + bp + cp^2)^{-1}$?
Answer:

$$(a + bp + cp^2)^{-1} = \frac{(a^2 - bc) + p(c^2 - ab) + p^2(b^2 - ac)}{a^3 + b^3 + c^3 - 3abc}$$

(d) Is $(a + bp + cp^2)^{-1} \in N$?
Answer: Yes, the multiplicative inverse is in N, since it can be written in the form $d + ep + fp^2$.

(e) Show that if a,b, and c are positive, an inverse always exists as long as they are not all equal (edited 2022-8-29). Answer: This is a tricky question. If you are familiar with enough classical statistics, you may know the **arithmetic--geometric mean equality** that for any nonnegative integers,

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \sqrt[n]{\prod_{k=1}^{n} x_k},$$

which implies

$$a^3 + b^3 + c^3 - 3\sqrt[3]{a^3b^3c^3} \geq 0.$$

This is equal to zero if and only if $a = b = c$. Thus that the inverse exists as long as a, b, and c are all non-negative, and not all zero. Note, however, that the inverse itself is not positive, since the square of the smallest component minus the product of the other two components will be negative.

(f) Find $(a + bp + cp^2)$ that is not itself zero and has no inverse.
Answer: There are infinitely many possible answers, but one is $1 - p + 0p^2$.

6. Find functions $x(t)$ and $y(t)$ such that

$$\lim_{t\to 0} x(t) = 0,$$
$$\lim_{t\to 0} y(t) = 0,$$
$$\lim_{t\to 0} x(t)^{y(t)} = 1/2,$$

Answer: We haven't discussed this kind of issue directly yet, but it's hear to point out that limits in two dimensions are more complicated than those in one dimension. Observe that

$$\lim_{t\to 0}(0)^{y(t)} = 0,$$

while

$$\lim_{t\to 0}(x(t))^0 = 1.$$

If we take $x(t) = t$, and $y(t) = -\ln(2)/\ln(t)$, then both $x(t) \to 0$ and $y(t) \to 0$ as $t \to 0$. But now

$$y(t)\ln(t) = \ln(1/2)$$
$$\ln\left(x(t)^{y(t)}\right) = \ln(1/2)$$
$$x(t)^{y(t)} = \frac{1}{2}.$$

This all implies that $f(x, y) = x^y$ is not only discontinuous at $f(0, 0)$, but also that that there are atleast three and actually infinitely many accumulation points of the function in the neighborhood of $(0, 0)$.