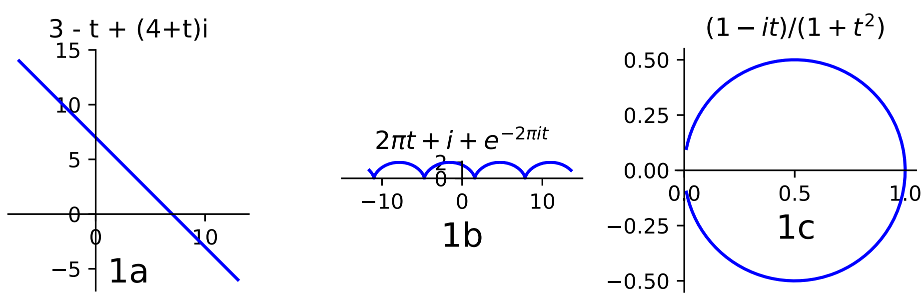
Homework 2, part A answers

1. $f(z) = \overline{z}$
2. $f(z) = 10 - iz/2$
3. $f(z) = e^{-\pi i/6}z$
4. $f(z) = -10 + 10i + z$
5. $f(z) = e^{3\pi i/4}z$
6. $f(z) = 10 + 10i - \overline{z}$
7. $f(z) = i\overline{z}$
8. $f(z) = -z$
9. $f(z) = 3z$
10. $f(z) = Re(z)/3 + iIm(z)$
11. too complicated -- a tricky nonlinear transformation

Homework 2, part B

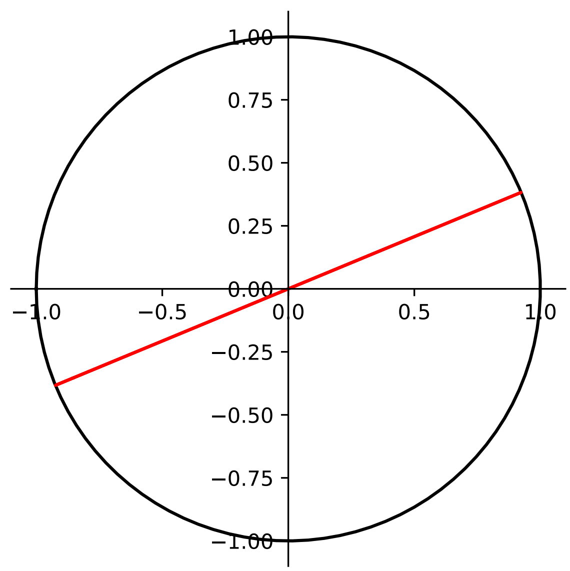


1.

2. So, we have a circle of radius 1/2 rolling inside a pipe or radius 1. Since the smaller circle (a wheel) rolls without slipping, and has half the perimeter of the outer circle, it must do two rotations as it rolls once around the circle. Also, the wheel will roll in the opposite orientation of the movement of it's center. **[TCR: corrected...]** After the wheel has rolled one full rotation, the initial tangent point will now be touching at the opposite side of the outer circle/pipe. Thus, the wheel will have completed 1/2 of a full rotation. Thus, in complex numbers, the equation of motion for a point p on the wheel will be

$$p(t) = \frac{1}{2}e^{(\alpha+\theta)i} + \frac{1}{2}e^{(\beta-\theta)i},$$

where α is the initial position of inner circle center, and β is the point's initial position relative to the circle center. The result, somewhat surprisingly, is a straight line -- a diameter of the circle, called the Tusi couple. This particular model is interesting in the history of science because of it's applications in medieval astronomy and because of it's property of turning circular motion into linear motion.



3. (a) The set of all points that are equidistant from i and $-i$ is the real axis $\text{Re}(z) = 0$.

   (b) If you remember, this is the classical definition of a hyperbola -- the set of all points for which the difference of their distances from their foci is constant. In this case, the hyperbola has foci at -2 and -5, with center $-7/2$ and asymptotes $y = \pm 2\sqrt{2}(x + 7/2)$. The slopes here are a little hard to calculate, but are given by the formula $m = \pm\sqrt{4(a/k)^2 - 1}$ where $a = 3/2$ (the distance from focus to center) and $k = \pm 1$.

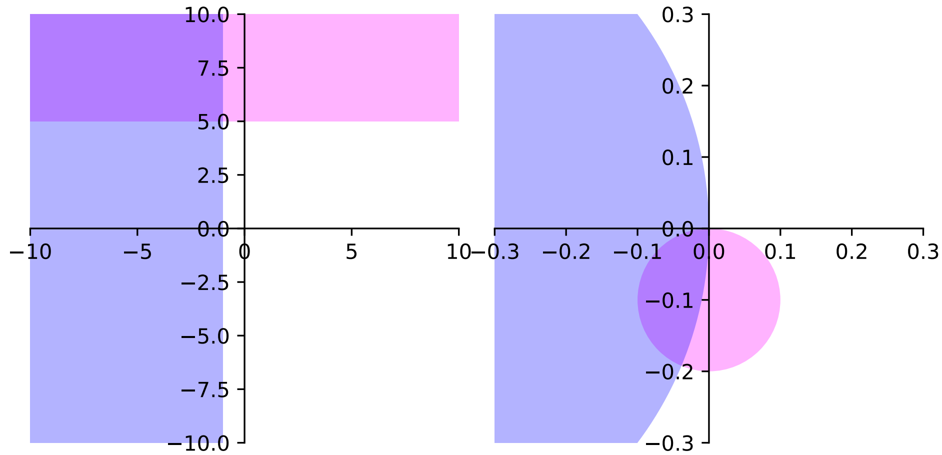
(c) $\text{Re}(i\bar{z}) = 4$ means $\text{Re}(i(x - iy)) = \text{Re}(y + ix) = y = 4$, so lines $z(x) = x + 4i$.

(d) This is a classic definition for an ellipse, as the set of all points whose sums of distances between 1 and -1 is 3.

4. $\{z \in \mathbb{C} : -\pi/3 \geq \text{Arg}\, z \geq -5\pi/6\}$.

5. $\{z \in \mathbb{C} : |z| > 1/2\}$.

6. The edges of the two regions are lines and are transformed to circles. The domains on the left are transformed to the domains on the right.



7. $\{z \in \mathbb{C} : 2\pi/3 \leq \text{Arg}\, z \leq \pi \text{ or } -\pi/3 \geq \text{Arg}\, z \leq -\pi\}$

8. $\{z \in \mathbb{C} : |z| < 4\}$

9. To find the square root, it is easiest to ask which curve, squared gives the boundary $\text{Re}(z^2) = -2$.

$$\text{Re}((x + iy)^2) = -2$$
$$x^2 - y^2 = -2$$
$$y^2 > x^2 + 2$$

There are two sides to this inequality -- one above the upper hyperbola, and one below the lower hyperbola. Both get mapped into the desired region. However, with the standard branch-cut along the negative real axis, only the halves of these areas with positive real part count, so

$$\{x + iy \in \mathbb{C} : x > 0 \text{ and } y^2 > x^2 + 2\}$$

10. To prove z^3 is continuous, consider

$$\begin{aligned}|(z+h)^3 - z^3| &= |3z^2h + 3zh^2 + h^3| \\ &= |3z^2 + 3zh + h^2||h| \\ &\leq (3|z|^2 + 3|z||h| + |h|^2)|h| \quad \text{by the triangle inequality.}\end{aligned}$$

Define $\delta(z, \epsilon) = \min(1, \epsilon/(3|z|^2 + 3|z| + 1))$. If $|h| < \delta(z, \epsilon)$, then $|h| < 1$ and $|h|^2 < 1$, so

$$\begin{aligned}(3|z|^2 + 3|z||h| + |h|^2)|h| &\leq (3|z|^2 + 3|z| + 1)|h| \\ &\leq (3|z|^2 + 3|z| + 1)\left(\frac{\epsilon}{3|z|^2 + 3|z| + 1}\right) \\ &= \epsilon.\end{aligned}$$

Thus, for every $\epsilon > 0$, there exists $\delta(z, \epsilon) = \min(1, \epsilon/(3|z|^2 + 3|z| + 1))$, such that $|h| < \delta(z, \epsilon)$ implies $|(z+h)^3 - z^3| < \epsilon$.

This is not the only bound one can use -- you might come up with something different.