Dangerous formulas of complex numbers

These formulas are reminders given without explicit conditions on when each applies -- the author is not responsible for the damages from formula mis-use.

The set of complex numbers can be represented

$$\mathbb{C} := \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = xI + yi : (x,y) \in \mathbb{R}^2 \right\},$$

where orthogonal basis vectors

$$\mathrm{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$\mathbb{C}$ is a vector space of matrices and matrix multiplication is bilinear, so $\mathbb{C}$ is an *algebra*. For a given $z \in \mathbb{C}$, we call $x = \mathrm{Re}(z)$ the real part, and $y = \mathrm{Im}(z)$ the imaginary part. If we abuse notation and leave out the I, we can write

$$z = x + iy = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$$

with abstract algebra relation $i^2 = -1$.
Complex conjugate: $\overline{z} := \mathrm{Re}(z) - i\,\mathrm{Im}(z)$

$$\mathrm{Re}(z) = (z + \overline{z})/2, \quad \mathrm{Im}(z) = (\overline{z} - z)i/2$$

Multiplication and Division:

$$\begin{aligned} cz &= (cx) + i(cy), \quad c \in \mathbb{R} \\ z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ &= z_2 z_1 \\ \mathrm{Im}(iz) &= \mathrm{Re}(z), \quad \mathrm{Re}(iz) = -\mathrm{Im}(z) \\ z^{-1} &= (x+iy)^{-1} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{z\overline{z}} \end{aligned}$$

Modulus: $|z| := \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{z\overline{z}} = r$

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Cardano's near-solution to cubic $p + qz - z^3 = 0$ is

$$\sqrt[3]{\sqrt{\left(\frac{p}{2}\right)^2 - \left(\frac{q}{3}\right)^3} + \frac{p}{2}} - \sqrt[3]{\sqrt{\left(\frac{p}{2}\right)^2 - \left(\frac{q}{3}\right)^3} - \frac{p}{2}}$$

Polar: $z = r(\cos\theta + i\sin\theta)$, radius r, argument θ

$$\arg z_1 + \arg z_2 = \arg(z_1 z_2)$$

de Moivre's formula: For $n \in \mathbb{N}$,

$$(\cos(t) + i\sin(t))^n = \cos(nt) + i\sin(nt)$$

Complex exponential

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\tanh z = \sinh z / \cosh z, \quad \tan z = \sin z / \cos z$$

Euler's formula $\cos(t) + i\sin(t) = e^{it}$, $e^{i\pi} = -1$

$$\begin{aligned} e^z &= e^{\mathrm{Re}(z)} e^{i\,\mathrm{Im}(z)} \\ \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) &= (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\ (re^{i\theta})^{-1} &= e^{-i\theta}/r \end{aligned}$$

Reciprocal powers when $n \in \mathbb{N}$,

$$\begin{aligned} z = 1^{1/n} &\leftrightarrow z^n = 1 \\ &\leftrightarrow z \in \{e^{2\pi i k/n} : k = 0 \dots n-1\} \\ z = a^{1/n} &= (re^{i\theta})^{1/n} \in |r^{1/n}| e^{i\theta/n + 2\pi\mathbb{Z}/n} \end{aligned}$$

$\mathrm{Arg}(z)$: $-\pi < \mathrm{Arg}(z) \le \pi$, $\mathrm{Arg}\, z - \arg z \in 2\pi\mathbb{Z}$.

$$z = |z| e^{i\,\mathrm{Arg}\, z}$$

$$\log z = \ln|z| + i\arg z, \quad \mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

Analytic: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$
Solution of a linear system $a + sv = p, b + tw = p$ for complex p, given s and t are real unknowns,

$$p = \left(\mathrm{Im}(\overline{a}v)w - \mathrm{Im}(\overline{b}w)v\right) / \mathrm{Im}(v\overline{w})$$

Vector products:

$$\begin{aligned} z_1 z_2 &= (\overline{z_1} \cdot z_2) + i(\overline{z_1} \times z_2) \\ &= (\overline{z_1} \cdot z_2) + i(z_1 \cdot i\overline{z_2}) \\ &= (\overline{z_1} \cdot z_2) - i(\overline{z_1} \cdot iz_2) \\ z_1 \cdot z_2 &= \mathrm{Re}(\overline{z_1} z_2) = (\overline{z_1} z_2 + z_1 \overline{z_2})/2 \\ &= |z_1||z_2|\cos(\theta_2 - \theta_1) \\ z_1 \cdot iz_2 &= -(iz_1 \cdot z_2) \\ z_1 \times z_2 &= \mathrm{Im}(\overline{z_1} z_2) = x_1 y_2 - x_2 y_1 \\ &= |z_1||z_2|\sin(\theta_2 - \theta_1) = \mathrm{Area}(z_1, z_2) \end{aligned}$$

$$\text{Polygon Area} = \frac{1}{2}\mathrm{Im}\left(\sum_{i=1}^{n} \overline{z_{i-1}} z_i\right), \ z_0 = z_n$$

Complex analysis: $\lim_{z \to 0} f(z) = w \leftrightarrow$

$$(\forall \epsilon > 0 \exists \delta > 0 : 0 < |z| < \delta \to |f(z) - w| < \epsilon)$$

$$\begin{aligned} f'(z) &:= \lim_{dz \to 0} \frac{f(z) - f(z - dz)}{dz} \\ (f(z) + g(z))' &= f(z)' + g(z)' \\ (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z) \\ f(g(z))' &= f'(g(z))g'(z) \\ (z^n)' &= nz^{n-1}, \ n \ne -1 \\ &\nexists\, (\overline{z})' \end{aligned}$$

Cauchy-Riemann eqs: When $f(z) = u(z) + iv(z)$,

$$u_x = v_y, \quad u_y = -v_x$$

$$f' = u_x + iv_x = u_x - iu_y = v_y + iv_x$$

$$\nabla u \cdot \nabla v = 0$$

$$u_{xx} = v_{xy} = -u_{yy}$$

$$v_{xx} = -u_{xy} = -v_{yy}$$

$f(z)$ is **holomorphic** in open neighborhood $U \subset \mathbb{C}$ when $\forall z \in U, \exists f'(z)$.

2D Vector calculus: Let Ω be a simply connected oriented closed subset of the 2-D plane, with smooth boundary $\partial\Omega$ parameterized by arclength s with normal unit vector $\hat{n}$ and tangent unit vector $\hat{T}$, and let $F(x,y) = [u(x,y), v(x,y)]$ be a continuous vector field with continuous first derivatives on Ω.

$$\text{Gradient } \nabla F = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

$$\text{Divergence } \nabla \cdot F = u_x + v_y$$

$$\text{Curl } \nabla \times F = v_x - u_y$$

Green's theorem in 2-d for : Inflows from internal sources equal outflows through the boundary.

$$\iint\limits_{\Omega} \nabla \cdot F d\Omega = \oint\limits_{\partial\Omega} F \times d(\partial\Omega) = \oint\limits_{\partial\Omega} F \cdot \hat{n} ds$$

Stoke's theorem in 2-d: Whatever circulation is created inside a domain must be observed as circulation on the boundary.

$$\iint\limits_{\Omega} \nabla \times F d\Omega = \oint\limits_{\partial\Omega} F \cdot d(\partial\Omega) = \oint\limits_{\partial\Omega} F \cdot \hat{T} ds$$

A holomorphic function $\Phi(z) = \phi(z) + i\psi(z)$ is also a complex potential where $\phi(z)$ is the electrostatic potential, $\psi(z)$ is the electrostatic flux, and both are harmonic.

$$\nabla\phi = \overline{\Phi'}, \quad \nabla\psi = i\,\overline{\Phi'}, \quad \nabla^2\phi = \nabla^2\psi = 0$$

$$\text{Curtain area: } \int\limits_{\Gamma} u(x,y) ds = \int\limits_{\Gamma} u(z)|dz|$$

$$\text{Work: } \int\limits_{\Gamma} u(x,y)dx + v(x,y)dy = \int\limits_{\Gamma} f(z) \cdot dz$$

x and y line integrals:

$$\int\limits_{\Gamma} u(x,y)dx = \text{Re} \int\limits_{\Gamma} u(z)\,\text{Re}(dz)$$

$$\int\limits_{\Gamma} u(x,y)dy = \text{Im} \int\limits_{\Gamma} u(z)\,\text{Im}(dz)$$

Complex integral:

$$\int\limits_{C} f(z)dz = \int\limits_{\Gamma} \overline{f}(z) \cdot dz + i \int\limits_{\Gamma} i\,\overline{f}(z) \cdot dz$$

$$= \int\limits_{C} u\,dx - \int\limits_{C} v\,dy + i \int\limits_{C} v\,dx + i \int\limits_{C} u\,dy$$

Polya's interpretation:

$$\oint\limits_{\partial\Omega} f(z)dz = \iint\limits_{\Omega} \left(\nabla \times \overline{f} + i\nabla \cdot \overline{f}\right)\, (dx \wedge dy)$$

ML-bound theorem:

$$\left| \int\limits_{\Gamma} f(z)dz \right| \leq |\Gamma| \max_{z\in\Gamma} |f(z)|$$

Fundamental theorem of complex calculus:

$$\frac{d}{dz} \int\limits_{a}^{z} f(w)dw = f(z)$$

.

Cauchy's theorem:

$$\oint\limits_{\partial\Omega} f(z)dz = 0$$

Cauchy's integral formula:

$$\oint\limits_{\partial\Omega} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Cauchy's extended integral formula:

$$\oint\limits_{\partial\Omega} \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} \frac{d^n f(a)}{dz^n}$$

Taylor series for functions holomorphic at a:

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \frac{d^n f(a)}{dz^n} \right) (z-a)^n$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\oint\limits_{\partial\Omega} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n$$

Laurent series at singularity a:

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n$$

Residue at a pole of order n

$$\text{Res}\left[\frac{f(z)}{(z-a)^n}, a \right] = \frac{1}{(n-1)!} \frac{d^{n-1} f(a)}{dz^{n-1}}$$

Residue theorem for Meromorphic functions:

$$\oint\limits_{\partial\Omega} f(z)dz = 2\pi i \sum_{a\in\Omega} \text{Res}\,[f,a]$$