Name:________________________________ Math 421, Quiz 3

Instructions: Clearly answer each of the questions below. Remember to check the back side. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers.

1. What are the Cauchy--Riemann conditions?

   For any holomorphic function $u(x, y) + iv(x, y)$, the partial derivatives of u and v always satisfy the partial differential equations $u_x = v_y$, $u_y = -v_x$

2. Use the Cauchy--Riemann conditions to directly test if the following is holomorphic.

$$e^x (\cos y + i \sin y).$$

   Answer: Here, $u(x, y) = e^x \cos y$, and $v(x, y) = e^x \sin y$, so

$$\begin{aligned} u_x &= e^x \cos y, \\ u_y &= -e^x \sin y, \\ v_x &= e^x \sin y, \\ v_y &= e^x \cos y. \end{aligned}$$

   By inspection, we see the Cauchy--Riemann conditions are satisfied. But remember this is only a necessary condition for holomorphicity. Fortunately, the partial derivatives are also continuous real functions everywhere in the complex plane, so the sufficient conditions are satisfied and this expression is an entire function -- holomorphic for all of the complex plane.

3. Show that $f(z) = \exp(-1/z^2)$ is not a continuous complex function at $z = 0$, even though it is a continuous real function.

   Answer: If $z \in \mathbb{R}$, then as $z \to \infty$, $f(z) \to \exp(-\infty) = 0$, so as long as we define $f(0) = 0$, $f(z)$ is continuous at $z = 0$. But now let's consider $z = \epsilon e^{i\pi/4}$ as $\epsilon \to 0$, so $z^2 = i\epsilon^2$.

$$\begin{aligned} f(\epsilon e^{i\pi/4}) &= \exp\left(-1/(i\epsilon^2)\right) \\ &= \exp\left(-i/\epsilon^2\right) \end{aligned}$$

   As $\epsilon \to 0$ this has no limit, continuing to oscillate. Thus, there can not be a unique limiting point for $f(z)$ as $z \to 0$.

4. Discuss the differentiability of $f(z) = z - \overline{z}$.

Answer: From the definition of a complex derivative,

$$\begin{aligned} \frac{((z+h) - (\overline{z+h})) - (z - \overline{z})}{h} &= \frac{h - \overline{h}}{h} \\ &= 1 - e^{-2i \operatorname{Arg} h} \end{aligned}$$

This limit depends on the direction from which h approaches 0, and is not unique for any value of z. Thus, the complex derivative of $f(z)$ does not exist anywhere in the complex plane $\mathbb{C}$.

You could also show that $f(z) = 2i\,\mathrm{Im}(z)$, so $u(x+iy) = 0$ and $v(x+iy) = 2y$. Then $u_x = 0$ but $v_y = 2$, so the Cauchy--Riemann conditions aren't satisfied anywhere either.