Lecture 2
Historical Notes

One of the major problems in which scientists of antiquity were involved was the study of planetary motions. In particular, predicting the precise time at which a lunar eclipse occurs was a matter of considerable prestige and a great opportunity for an astronomer to demonstrate his skills. This event had great religious significance, and rites and sacrifices were performed. To make an accurate prediction, it was necessary to find the true instantaneous motion of the moon at a particular point of time. In this connection we can trace back as far as, Bhaskara II (486AD), who conceived the differentiation of the function $\sin t$. He was also aware that a variable attains its maximum value at the point where the differential vanishes. The roots of the mean value theorem were also known to him. The idea of using integral calculus to find the value of π and the areas of curved surfaces and the volumes was also known to Bhaskara II. Later Madhava (1340–1429AD) developed the limit passage to infinity, which is the kernel of modern classical analysis. Thus, the beginning of calculus goes back at least 12 centuries before the phenomenal development of modern mathematics that occurred in Europe around the time of Newton and Leibniz. This raises doubts about prevailing theories that, in spite of so much information being known hundreds of years before Newton and Leibniz, scientists never came across differential equations. The information which historians have recorded is as follows:

The founder of the differential calculus, Newton, also laid the foundation stone of DEs, then known as fluxional equations. Some of the first-order DEs treated by him in the year 1671 were

$$y' = 1 - 3x + y + x^2 + xy \tag{2.1}$$

$$3x^2 - 2ax + ay - 3y^2y' + axy' = 0 \tag{2.2}$$

$$y' = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}, \quad \text{etc.} \tag{2.3}$$

$$\begin{aligned} y' \;=\; & -3x + 3xy + y^2 - xy^2 + y^3 - xy^3 + y^4 - xy^4 \\ & +6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4, \quad \text{etc.} \end{aligned} \tag{2.4}$$

He also classified first-order DEs into three classes: the first class was composed of those equations in which y' is a function of only one variable, x alone or y alone, e.g.,

$$y' = f(x), \quad y' = f(y); \tag{2.5}$$

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the second class embraced those equations in which y' is a function of both x and y, i.e., (1.9); and the third is made up of partial DEs of the first order.

About five years later, in 1676, another independent inventor of calculus, Leibniz, coined the term *differential equation* to denote a relationship between the differentials dx and dy of two variables x and y. This was in connection with the study of geometrical problems such as the inverse tangent problem, i.e., finding a curve whose tangent satisfies certain conditions. For instance, if the distance between any point $P(x, y)$ on the curve $y(x)$ and the point where the tangent at P crosses the axis of x (length of the tangent) is a constant a, then y should satisfy first-order nonlinear DE

$$y' = -\frac{y}{\sqrt{a^2 - y^2}}. \tag{2.6}$$

In 1691, he implicitly used the method of separation of variables to show that the DEs of the form

$$y\frac{dx}{dy} = X(x)Y(y) \tag{2.7}$$

can be reduced to quadratures. One year later he integrated linear homogeneous first-order DEs, and soon afterward nonhomogeneous linear first-order DEs.

Among the devoted followers of Leibniz were the brothers James and John Bernoulli, who played a significant part in the development of the theory of DEs and the use of such equations in the solution of physical problems. In 1690, James Bernoulli showed that the problem of determining the isochrone, i.e., the curve in a vertical plane such that a particle will slide from any point on the curve to its lowest point in exactly the same time, is equivalent to that of solving a first-order nonlinear DE

$$dy(b^2y - a^3)^{1/2} = dx\ a^{3/2}. \tag{2.8}$$

Equation (2.8) expresses the equality of two differentials from which Bernoulli concluded the equality of the integrals of the two members of the equation and used the word *integral* for the first time on record.

In 1696 John Bernoulli invited the brightest mathematicians of the world (Europe) to solve the brachistochrone (quickest descent) problem: to find the curve connecting two points A and B that do not lie on a vertical line and possessing the property that a moving particle slides down the curve from A to B in the shortest time, ignoring friction and resistance of the medium. In order to solve this problem, one year later John Bernoulli imagined thin layers of homogeneous media, he knew from optics (Fermat's principle) that a light ray with speed ν obeying the law of Snellius,

$$\sin\alpha = K\nu,$$

passes through in the shortest time. Since the speed is known to be proportional to the square root of the fallen height, he obtained by passing through thinner and thinner layers

$$\sin\alpha \;=\; \frac{1}{\sqrt{1+y'^2}} \;=\; K\sqrt{2g(y-h)}, \tag{2.9}$$

a differential equation of the first order. Among others who also solved the brachistochrone problem are James Bernoulli, Leibniz, Newton, and L'Hospital.

The term "separation of variables" is essentially due to John Bernoulli; he also circumvented dx/x, not well understood at that time, by first applying an integrating factor. However, the discovery of integrating factors proved almost as troublesome as solving a DE.

The problem of finding the general solution of what is now called *Bernoulli's equation*,

$$a\,dy \;=\; yp\,dx + bqy^n\,dx, \tag{2.10}$$

in which a and b are constants, and p and q are functions of x alone, was proposed by James Bernoulli in 1695 and solved by Leibniz and John Bernoulli by using different substitutions for the dependent variable y. Thus, the roots of the general tactic "change of the dependent variable" had already appeared in 1696–1697. The problem of determining the orthogonal trajectories of a one-parameter family of curves was also solved by John Bernoulli in 1698. And by the end of the 17th century most of the known methods of solving first-order DEs had been developed.

Numerous applications of the use of DEs in finding the solutions of geometric problems were made before 1720. Some of the DEs formulated in this way were of second or higher order; e.g., the ancient Greek's isoperimetric problem of finding the closed plane curve of given length that encloses the largest area led to a DE of third order. This third-order DE of James Bernoulli (1696) was reduced to one of the second order by John Bernoulli. In 1761 John Bernoulli reported the second-order DE

$$y'' \;=\; \frac{2y}{x^2} \tag{2.11}$$

to Leibniz, which gave rise to three types of curves—parabolas, hyperbolas, and a class of curves of the third order.

As early as 1712, Riccati considered the second-order DE

$$f(y, y', y'') \;=\; 0 \tag{2.12}$$

and treated y as an independent variable, $p\ (= y')$ as the dependent variable, and making use of the relationship $y'' = p\,dp/dy$, he converted the

DE (2.12) into the form

$$f\left(y, p, p\left(\frac{dp}{dy}\right)\right) \;=\; 0, \tag{2.13}$$

which is a first-order DE in p.

The particular DE

$$y' \;=\; p(x)y^2 + q(x)y + r(x) \tag{2.14}$$

christened by d'Alembert as the *Riccati equation* has been studied by a number of mathematicians, including several of the Bernoullis, Riccati himself, as well as his son Vincenzo. By 1723 at the latest, it was recognized that (2.14) cannot be solved in terms of elementary functions. However, later it was Euler who called attention to the fact that if a particular solution $y_1 = y_1(x)$ of (2.14) is known, then the substitution $y = y_1 + z^{-1}$ converts the Riccati equation into a first-order linear DE in z, which leads to its general solution. He also pointed out that if two particular solutions of (2.14) are known, then the general solution is expressible in terms of simple quadrature.

For the first time in 1715, Taylor unexpectedly noted the singular solutions of DEs. Later in 1734, a class of equations with interesting properties was found by the precocious mathematician Clairaut. He was motivated by the movement of a rectangular wedge, which led him to DEs of the form

$$y \;=\; xy' + f(y'). \tag{2.15}$$

In (2.15) the substitution $p = y'$, followed by differentiation of the terms of the equation with respect to x, will lead to a first-order DE in x, p and dp/dx. Its general solution $y = cx + f(c)$ is a collection of straight lines. The Clairaut DE has also a singular solution which in parametric form can be written as $x = -f'(t),\ y = f(t) - tf'(t)$. D'Alembert found the singular solution of the somewhat more general type of DE

$$y \;=\; xg(y') + f(y'), \tag{2.16}$$

which is known as *D'Alembert's equation*.

Starting from 1728, Euler contributed many important ideas to DEs: various methods of reduction of order, notion of an integrating factor, theory of linear equations of arbitrary order, power series solutions, and the discovery that a first-order nonlinear DE with square roots of quartics as coefficients, e.g.,

$$(1 - x^4)^{1/2}y' + (1 - y^4)^{1/2} \;=\; 0, \tag{2.17}$$

has an algebraic solution. Euler also invented the method of variation of parameters, which was elevated to a general procedure by Lagrange in

1774. Most of the modern theory of linear differential systems appears in D'Alembert's work of 1748, while the concept of adjoint equations was introduced by Lagrange in 1762.

The *Jacobi equation*

$$(a_1 + b_1x + c_1y)(xdy - ydx) - (a_2 + b_2x + c_2y)dy + (a_3 + b_3x + c_3y)dx \;=\; 0$$

in which the coefficients $a_i,\ b_i,\ c_i,\ i = 1,2,3$ are constants was studied in 1842, and is closely connected with the Bernoulli equation. Another important DE which was studied by Darboux in 1878 is

$$-Ldy + Mdx + N(xdy - ydx) \;=\; 0,$$

where $L,\ M,\ N$ are polynomials in x and y of maximum degree m.

Thus, in early stages mathematicians were engaged in formulating DEs and solving them, tacitly assuming that a solution always existed. The rigorous proof of the existence and uniqueness of solutions of the first-order initial value problem (1.9), (1.10) was first presented by Cauchy in his lectures of 1820–1830. The proof exhibits a theoretical means for constructing the solution to any desired degree of accuracy. He also extended his process to the systems of such initial value problems. In 1876, Lipschitz improved Cauchy's technique with a view toward making it more practical. In 1893, Picard developed an existence theory based on a different method of successive approximations, which is considered more constructive than that of Cauchy–Lipschitz. Other significant contributors to the method of successive approximations are Liouville (1838), Caqué (1864), Fuchs (1870), Peano (1888), and Bôcher (1902).

The pioneering work of Cauchy, Lipschitz, and Picard is of a qualitative nature. Instead of finding a solution explicitly, it provides sufficient conditions on the known quantities which ensure the existence of a solution. In the last hundred years this work has resulted in an extensive growth in the qualitative study of DEs. Besides existence and uniqueness results, additional sufficient conditions (rarely necessary) to analyze the properties of solutions, e.g., asymptotic behavior, oscillatory behavior, stability, etc., have also been carefully examined. Among other mathematicians who have contributed significantly in the development of the qualitative theory of DEs we would like to mention the names of R. Bellman, I. Bendixson, G. D. Birkhoff, L. Cesari, R. Conti, T. H. Gronwall, J. Hale, P. Hartman, E. Kamke, V. Lakshmikantham, J. LaSalle, S. Lefschetz, N. Levinson, A. Lyapunov, G. Peano, H. Poincáre, G. Sansone, B. Van der Pol, A. Wintner, and W. Walter.

Finally the last three significant stages of development in the theory of DEs, opened with the application of Lie's (1870–1880s) theory of continuous groups to DEs, particularly those of Hamilton–Jacobi dynamics;

Picard's attempt (1880) to construct for linear DEs an analog of the Galois theory of algebraic equations; and the theory, started in 1930s, that paralleled the modern development of abstract algebra. Thus, the theory of DEs has emerged as a major discipline of modern pure mathematics. Nevertheless, the study of DEs continues to contribute to the solutions of practical problems in almost every branch of science and technology, arts and social science, and medicine. In the last fifty years, some of these problems have led to the creation of various types of new DEs, some which are of current research interest.