Homework 1

Spring 2025 Math 511, Penn State UP

General ODEs (including Differential-algebraic equations

1. Find an ordinary differential equation of the form $F(x, y(x), y') = 0$ satisfied by every curve $y(x; a) = ax^2$. (your equation should not depend on a).

2. Find a differential equation $F(x, y, y') = 0$ satisfied by every tangent lines to the hyperbola $xy = 1$. Write your solution in the form of a Clairaut equation $xy' + g(y') = y$. For what domain of initial conditions do solutions to this equation not exist?

3. Find all line-solutions $y(x)$ of
$$y(x) = xy' + \frac{1}{1 + (y')^2}.$$

4. Find all smooth solutions $y(x)$ of $1 - 4y' + 3(y')^2 = y$, and determine where solutions do not exist.

5. Formally, we might define a weak derivative as of $f()$ at z as D if and only if $\forall \epsilon > 0$, $\forall \delta > 0$, $\exists x, y$ such that $|x - z| < \delta$ and $|y - z| < \delta$ and $|\frac{f(y)-f(x)}{y-x} - D| < \epsilon$. Show that $D = 1/2$ is a weak derivative of $f(x) = |x|$ at $x = 0$, but that $D = 2$ is not.

6. Find an example function for which (a) the set of weak derivatives is $\mathbb{R}$, (b) the set of weak derivatives is empty, (c) the set of weak derivatives is $[1, 2]$.

7. Another geometric application of ordinary differential equations is construction of orthogonal coordinate systems. Given a family of curves $y - x^2 = a$ for $x > 0$ and $y > 0$, (1) find a formula for $y'(x, y)$, (2) transform this to find the slope of an orthogonal curve at each point, and (3) integrate to find the orthogonal curve family.

8. Make a case for considering $y(x) = 1$ a solution of the ODE $(y')^2 = 1$. (Hint: Any continuous function compose of line segments with slopes of 1 or -1 will always have left and right derivatives that solve this equation.)

Explicit first-order equations

9. Find a general solution to the follow $t > 0$,
$$t^2 \frac{dx}{dt} = xt + 3x^2.$$

10. In class, we saw that quadratic forms $\alpha x^2 + \beta xy + \gamma y^2 = C$ could solve certain 1st-order ODEs. Generalize this method to solve
$$y' = \frac{4x - 3y + 5}{3x + 4y - 10}.$$

Exact equations

11. Show that if $M(x, y) + N(x, y)y' = 0$ is an exact equation, then the "inhomogeneous" case
$$M(x, y) + N(x, y)y' = P(x)$$
can also be written as an exact equation.

12. Show that separable equations are a strict subset of exact equations.

13. Find a function $f(u,v)$ such that $f(x,y(x)) = C$ implicitly defines a solution of

$$y' = \frac{x^3 - 3xy^2}{3x^2y - y^3}$$

14. Solve

$$y' = \frac{y\left(5 + 2xy^2\right)}{x\left(-5 + xy^2\right)}$$

15. Given an equation of the form $y' = M(x,y)/N(x,y)$, and hypothesizing the integrating factor ansatz $\mu(x,y) = f(y)$ such that $f(y)N(x,y)y' - f(y)M(x,y) = 0$ is an exact equation, what should the general form of $f(y)$ be?