Homework 2

Spring 2025 Math 511, Penn State UP

Picard Iteration

1. Using the Picard iteration formula, calculate the first three iterations for approximating the solution of

$$y' = 1 + (\exp(-x) - 1)y, \; y(0) = 2.$$

2. Using the Picard iteration formula, calculate the first three iterations for approximating the solution of $\vec{y}(t)$ when $\vec{y}(0) = [a, b]$ and

$$\frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = ty_1.$$

Lie Groups

3. Show that the 1-parameter family of transformations $F_t(x, y) = (tx, y + xty$ does **not** form a group under composition for $t \in \mathbb{R}$.

4. Show that the 1-parameter family of transformations $F_t(x, y) = (x + t, y + xt^2)$ does **not** form a group under composition for $t \in \mathbb{R}$.

5. Prove that the 1-parameter family of transformations $F_t(x, y) = (x + t, y + ty/x)$ **does** form a group under composition for $t \in \mathbb{R}$.

Homogeneous Linear Systems

6. Find the general solution of the system $y' = Ay$ where

$$A = \begin{bmatrix} 206 & 12 & -54 \\ 12 & 98 & 48 \\ -54 & 48 & 241 \end{bmatrix}$$

in terms of eigenvectors and eigenvalues.

7. Describe how the behavior of solutions to the linear system

$$\dot{x} = acx + bcy, \quad \dot{y} = adx + bdy$$

depend on the parameter values a,b,c, d (all of which are real numbers).

8. The theory of linear difference equations has allot in common with linear differential equations. Consider $x_{t+1} = Ax_t$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Find a set of conditions on the components of A sufficient to imply $\lim_{t\to\infty} x_t = 0$. (You can use the coefficients of the characteristic polynomial, but not the eigenvalues.)

9. Given cubic

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

derive conditions for all roots x to have negative real part.

Linear Inhomogeneous equations

10. Find the Green function for the 2nd-order linear equation

$$y'' - a^2 y = h(x)$$

11. Given a general 2nd-order variable-coefficient equation $y'' + p(x)y' + q(x)y = h(x)$, where $y_1(x)$ and $y_2(x)$ are linearly independent and both annihilated by the linear operator $D^2 + p(x)D + q(x)$, find the general form for the Green function, such that

$$y(x) = \int_0^t g(x,u)h(u)du.$$

12. Can you generalize the method above to 3 third-order variable-coefficient equations?

13. Find the lowest-order linear annihilation operator A for $x\sin(3x)$ such that $(A)(x\sin 3x) = 0$. (Here the order of the operator is the number of derivatives it requires.)

Answer: We find the annihilator $A = (D^2+9)^2$. Then, inverting our operator, we find

$$\frac{1}{D^2+1} = \frac{-1}{64}\frac{D^2+17}{1-(D^2+9)^2/64}$$

Applied, to our inhomogeneity, we find the general solution

$$y(x) = -\frac{3\cos(3x)}{32} - \frac{x\sin(3x)}{8} + C_1\sin(x) + C_2\cos(x). \tag{1}$$

14. Use the method of operator inversion to solve $y'' + y = x\sin(3x)$.

15. Find a continuous group under which the family of solutions of the linear homogeneous equidimensional equations are invariant

$$0 = a_0 y + a_1 x y' \dots a_n x^n y^{(n)}$$

16. A first-order equidimensional linear operator has the form $L = (xD - a)$. Show that any two such operators commute.