Homework 4

Spring 2025 Math 511, Penn State UP

1. For the damped pendulum equation $y'' = -\sin(y) - ky'$, the total energy is only a weak Lyapunov function. Find a strong Lyapunov function of the form $L(y, y') = y^2 + ayy' + b(y')^2$ that establishes the asymptotic stability of the steady-state solution $y = y' = 0$ for all $k > 0$.

2. Consider the following system of ODE's that describes a 1-resource competition model:

$$\dot{r} = n - axr - byr,$$
$$\dot{x} = axr - dx$$
$$\dot{y} = byr - fy.$$

We will make the common assumption that all parameters have positive values.

(a) Use the assumption that $r = n/(ax + by)$ to reduce this system from 3 equations to two equations. (This is called a quasi-steady-state approximation. We will talk about it more in the future.)

(b) Show that the system from part (a) can not have limit-cycle solutions in the positive quadrant ($x \geq 0$, $y \geq 0$) by finding a Dulac function $B(x, y)$ that allows us to apply the Bendixson--Dulac negative criteria.

(c) Determine the ω-limits for all initial conditions with atleast one positive coordinate, assuming (without loss of generality) that $a/d > b/f$.

3. One way to prove a stable limit-cycle exists within a simply-connected (no holes) closed and bounded domain is to show that there (1) the vectors point inward at every point on the boundary, and (2) that all the stationary solutions inside the domain are unstable foci or nodes. Then the Poincare--Bendixson theorem implies that each nonstationary initial conditions in the domain must have a limit cycle as its ω-limit set.

Show that

$$\dot{x}_1 = a - x_1 + x_1^2 x_2$$
$$\dot{x}_2 = b - x_1^2 x_2$$

has a limit cycle inside the quadrilateral

$$[(a + b + b/a^2 + \delta, 0), (a, 0), (a, b/a^2), (a + b + \delta, b/a^2)],$$

if δ is a small positive number, $a = 0.1$, and $b = 0.3$.