Provisioning of public health can be designed to anticipate public policy responses

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Abstract

Public health policies can elicit strong responses from individuals. These responses can promote, reduce and even reverse the expected benefits of the policies. Therefore, projections of individual responses to policy can be important ingredients in policy design. Yet our foresight of individual responses to public health investment remains limited. This paper formulates a population game describing the prevention of infectious disease transmission when community health depends on the interactions of individual and public investments. We compare three common relationships between public and individual investments and explain how each relationship alters policy responses and health outcomes. Our methods illustrate how identifying system interactions between nature and society can help us anticipate policy responses.

Keywords: epidemiological games, infectious disease, community health, policy resistance, policy reinforcement, health commons

1 Introduction

Thirteen million deaths occur every year from preventable infectious diseases [35]. This is a disappointing number, given the widespread optimism during the 1960s that we would soon conquer infectious disease [20, 25]. These deaths occur despite notable advances in infectious disease management, including vaccination programs, well-developed infrastructure, and improved hygiene practices and medical care. Given these advances, why does infectious disease remain a problem?

One reason infectious-disease management practices have not been uniformly successful is that they do not operate in a vacuum -- they are part of a larger health commons including social, economic, environmental, and ecological pressures that can impede our management efforts. We introduce the term ‘‘health commons’’ as a parallel to common-pool resource management problems. A health-commons defines to shared community space where the actions of members and groups can impact the health of both themselves and those with whom they share the space. Although health itself is not a good or service under standard economic definitions, it naturally subsumes pressures from pollution, nutrition, and disease, which have long been associated with tragedies of the commons [21]. To be fully understood, public health practices must be contextualized as part of a health commons, including the full variety of system pressures and their feedbacks. In particular, explanations of policy efficacy must account for feedbacks from human behavior [14]. Identifying how management practices change the pressures on individuals and how individuals react to changing pressures is often critical to explaining how effectively these practices will improve the shared health within a community [32].

There is a growing body of literature on quantifying the impact of human behavior in health commons [16]. Many efforts to study human behavior in epidemiology center on weighing the effectiveness of various centrally coordinated policies [5, 22]. Gersovitz [17] and Gersovitz and Hammer [18] discussed the economic aspects of the control of infectious diseases, investigating the decisions of social planners and the representative decision-making agents who directly control preventive and therapeutic efforts. Research articles using game theory study individuals’ responses to the incentives associated with vaccination [15, 6, 3, 2, 9, 11, 29, 30] and transmission reduction [8, 28, 31, 19, 10]. These advances indicate that game theory gives a way to analyze how people may respond to economic and epidemiological pressures.

As with natural resource management [27], the nature of a health commons is shaped by the actions of both individuals and public institutions. Global efforts to reduce childhood diarrhea provide one example [33]. The achievement of goals like slowing transmission of childhood diarrhea can be influenced by diverse investments, with some traditionally provided by governments while other investments are provided by individuals. Governments can invest in sanitation infrastructure and water treatment facilities, while individuals can invest in washing their hands and treating their own water supplies. Governments or individuals may invest independently to reduce disease transmission, but it is also possible that they invest concurrently. (e.g., governments may invest in sanitation and people may wash their hands more.) This last possibility brings up questions such as how investments in sanitation influence hand-washing. Sometimes people respond to public investment by reducing their own investment, a phenomenon called ‘‘policy resistance’’ [32]. Other times, people respond to public investment by increasing their own investment. We call this ‘‘policy reinforcement’’. These reactions may have un-anticipated consequences, and public health management should take these feedbacks into account when deciding on intervention options [1].

Our aim here is to study the interaction between public and individual investment in a health commons as mediated by infectious disease dynamics. Our findings provide insights into effective ways of leveraging public and individual investments to reduce disease transmission and allow us to anticipate negative outcomes. To quantify public health consequences from government and individual interventions, we first need to identify how interventions interact with contextual processes and disease transmission cycles within a larger systems-wide theory of community health and infectious disease spread. We focus on interventions that will reduce the public's risk of acquiring infection (e.g., hygiene and social distancing). Epidemiology will be described using an SIS model at steady-state. We first analyze individuals' actions. Game equilibria provide us with an idealized individual response to the costs of infection and prevention. We use geometric methods to identify a general bound for the equilibria. Then, we discuss the relationships between individual and public investment, and obtain geometric results on when policy resistance or reinforcement arise in response to policy changes. Several examples are solved to illustrate our results.

2 Model formulation

Rather than building a complicated model in attempt to capture all aspects of the infectious disease health commons, we will make a simple model with the goal of revealing fundamental principles that may apply more generally. Our general method follows the approach of Reluga and Galvani [30]. We will use a susceptible-infected-susceptible (SIS) model to describe the population-dynamics of a non-immunizing bacterial infection [23], leading to a population game that is an extension of one first introduced by Chen [8] and similar to Gersovitz [17]. We will limit ourselves to analysis to situations when disease dynamics are near equilibrium with all parameters constant in time. Generalizations time-dependent effects are considered in the Discussion. Our methods are just as appropriate for more complicated disease theories, but we apply them here to the simplest of scenarios so that the methods may be illustrated with a relatively complete set of results. For a broader consideration of the potential complications of disease transmission, see Reluga and Galvani [30] and the citations there-in.

Consider a community with N individuals, where N is large. At any given time, each individual may be susceptible to infection, or infected. Let S represent the number of susceptible individuals and I represent the number of infected individuals, with $N = S + I$. The changes in the number of susceptible and infected individuals per unit time are governed by a system of differential equations

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sigma(\bar{c}_s, c_t)\lambda S + \gamma I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \sigma(\bar{c}_s, c_t)\lambda S - \gamma I, \tag{1}$$

where λ is referred to as the infection pressure, which is the rate at which susceptible individuals acquire infection from exposure to infected individuals when there are no extra interventions either by individuals to protect themselves or by the government to protect the public. We use the standard-incidence hypothesis that the infection pressure is the transmission rate β times the fraction of individuals infected under the assumption of constant population size. Therefore $\lambda := \beta I/N$. Infected individuals recover at a rate γ and since they have not gained any immunity, they return to a susceptible state.

The disease burden can be reduced by various interventions. Here, we focus only on interventions that protect people by reducing their risk of infection. The function $\sigma(\cdot,\cdot)$ in Eq. (1), called the relative exposure rate, includes effects from both individuals and government. Individuals may change their personal behaviors (e.g., food preparation, hand washing, reduced social contact) in return for reductions in their relative exposure rate. In addition, government may invest in public health infrastructure (sanitation, water supply, nutrition, education, advertising, etc.) at per capita rate c_t to reduce people's exposures to infection. The relative exposure rate $\sigma(c_s, c_t)$ of an individual adopting behavior change c_s under public investment rate c_t is the risk of infection per unit time of an individual, relative to the infection pressure. The typical relative exposure rate is $\sigma(\bar{c}_s, c_t)$, depending on the typical behavior change $\bar{c}_s$ in the population. At the population-scale, only the typical behavior matters, so the prevalence predicted by System (1) depend on $\bar{c}_s$ and not on any individual choice c_s.

The typical relative exposure rate function $\sigma(\bar{c}_s, c_t)$ encapsulates the relationship between financial inputs and the real-world events that change the rate individuals come into contact with an infectious agent. By assumption, we will pick $\sigma(0, 0) = 1$ so that β is the baseline transmission rate when no extra actions are taken. The baseline may already account for some pre-existing positive investments by individuals and governments in general health practices. $\sigma(c_s, c_t)$ is decreasing in both c_s and c_t while staying positive. The shape of $\sigma(\bar{c}_s, c_t)$ depends on the effectiveness and implementation details of public investment and individual behavior. If intervention is ineffective, σ will decrease slowly as investment is increased. If an intervention is efficient, σ may decrease quickly. If interventions can eliminate all means of exposure, then $\lim_{c_t \to \infty} \sigma(\overline{c}_s, c_t) = 0$, but if some means of exposure are unaffected by government investment, we may have $\lim_{c_t \to \infty} \sigma(\overline{c}_s, c_t) > 0$. If large maintenance expenditures are required from government before any exposure-reductions are realized, σ may exhibit a threshold-behavior, with a shallow slope for small c_t, followed by a steep slope near the maintenance threshold.

We will see that this shape controls some of the most interesting features of social planning. While our theorems make use of only general geometric properties, we will use a number of specific functional forms in our figures for illustration. Relative exposure rates of the form $\exp(-a_s c_s - a_t c_t)$ model interventions that act to reduce exposures multiplicatively, such as increasing the layers of filtering in water use. Hill functions can be used in forms like $1 - f(c_s) + f(c_s)/(1 + (a_t c_t)^n)$ to capture threshold effects in government investment, where minimal capital costs for construction and maintenance must be met before there is any effectiveness.

System (1) describes an endemic-disease scenario, and can be used to estimate people's risk of infection. To determine the stationary infection pressure $\widetilde{\lambda}$, we perform an equilibrium analysis. The only two stationary solutions are the endemic solution $(\widetilde{S}, \widetilde{I}) = (N\gamma/(\beta\sigma(\overline{c}_s, c_t)), N(1 - \gamma/(\beta\sigma(\overline{c}_s, c_t))))$ and the disease-free solution $(S, I) = (N, 0)$. The disease-free stationary solution is globally attractive if the effective reproductive number $\sigma(\overline{c}_s, c_t)\mathcal{R}_0 := \sigma(\overline{c}_s, c_t)\beta/\gamma \leq 1$. If $\sigma(\overline{c}_s, c_t)\mathcal{R}_0 > 1$, then the disease-free stationary solution is unstable and the unique endemic stationary solution is globally attracting. When restricting our analysis to stable steady-states, the infection pressure

$$\widetilde{\lambda}(\overline{c}_s, c_t) := \max\left\{0, \beta - \frac{\gamma}{\sigma(\overline{c}_s, c_t)}\right\}. \tag{2}$$

To describe human behavior, we employ a population-game approach where people make choices that maximize the expected utility of their returns now and in the future assuming perfect information about their own state and the worlds state is available to them. The expected utility can be calculated based on knowledge of the rates of increase or decrease in utility per unit time from various sources, aka ''utility gains''. Utility gains to an individual with income j are given by $u(j)$, an increasing function with diminishing returns. In a sustainable scenario, a government with a balanced budget acquires its resources through taxation, so we assume public investment always incurs income losses to individuals. After incorporating the on-going costs of public investment, utility gains are reduced to $u(j - c_t)$. While the process of revenue collection can itself alter behavior, we leave this as an open topic and instead focus on the consequences of these income reductions. For the population game we are studying, any reduction of risk of infection comes at the expense of a partial loss of utility gains. The behavior choice parameter c_s should be interpreted as the rate an individual invests some of their utility gains in self-protection, while $\overline{c}_s$ represents the typical investment rate. Since this investment has no benefit for people already infected, it should only be made while susceptible. The vector of net utility gains from residence in each state per unit time under these conditions is $\mathbf{v} := \left[u(j-c_t)-c_s, u(j-c_t)-c_i\right]$, where c_i is the cost rate of infection. To calculate the total expected utility of such investments for an individual in a population with typical investment rate $\overline{c}_s$ and public investment rate c_t, we apply Markov decision process theory [24]. The probabilities $\mathbf{p}(t)$ that an individual is in the susceptible or infected states at time t are determined by a Markov process according to

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{Q}\mathbf{p} \quad \text{where} \quad \mathbf{Q} := \begin{bmatrix} -\sigma(c_s,c_t)\lambda & \gamma \\ \sigma(c_s,c_t)\lambda & -\gamma \end{bmatrix}. \tag{3}$$

When the population dynamics are near steady-state, the expected utility of the investment c_s to an individual initially in the susceptible state, $\mathbf{p}_0 = [1,0]^T$, is

$$U(c_s,\overline{c}_s;c_t) := \int_0^\infty e^{-ht}\mathbf{v}\cdot\mathbf{p}(t)\mathrm{d}t = \int_0^\infty e^{-ht}\mathbf{v}e^{\widetilde{\mathbf{Q}}t}\mathbf{p}_0\mathrm{d}t \tag{4}$$

where h is the rate of discounting of future returns and $\widetilde{\mathbf{Q}}$ is the transition matrix evaluated at steady-state. This simplifies (see Appendix A for the derivation) down to

$$U(c_s,\overline{c}_s;c_t) = \mathbf{v}\left(h\mathbf{I}-\widetilde{\mathbf{Q}}\right)^{-1}\mathbf{p}_0 = \frac{u(j-c_t)}{h} - \frac{(h+\gamma)c_s + \widetilde{\lambda}(\overline{c}_s,c_t)\sigma\left(c_s,c_t\right)c_i}{h\left[h+\gamma+\widetilde{\lambda}(\overline{c}_s,c_t)\sigma\left(c_s,c_t\right)\right]} \tag{5}$$

with $\widetilde{\lambda}$ representing the infection pressure when Eq. (1) is at steady-state. We can think of the expected utility as the sum of the differences between the utility gains and the costs associated with disease prevention and infection, both discounted over time.

3 Population Game Analysis

The aim of this section is to develop basic equilibrium results for our population game. Some of these results will be generalizations of those of Chen [8]. We will start by looking for the best response for an individual player's investment that maximizes utility. The determination of the best response is shown to depend on the shape of the relative exposure rate function σ. Based on the basic properties of the best response, we can show that when $\sigma(c_s,c_t)$ is convex in c_s, there exists a unique game equilibrium. We will calculate this equilibrium for a given $\sigma(\cdot,\cdot)$. Furthermore, for any relative exposure rate convex in c_s, we can bound the equilibrium strategy.

3.1 Best Responses of Individuals

The first step in analyzing a population game is identifying the best response for an individual player whose behavior may differ from the typical behavior. This allows us to confirm a number of intuitive results, including the expectation that individuals will never choose to invest more in preventing disease than the disease itself costs.

An individual's best response c_s^B maximizes the utility of their investment, $c_s^B := \operatorname{argmax}_{c_s \geq 0} U(c_s)$. The best response either occurs at the boundary ($c_s^B = 0$) or is chosen so that the marginal cost of preventive investment equals the marginal benefit of less frequent infection. Differentiating U with respect to c_s, and noting that $u(j - c_t)$ and $\widetilde{\lambda}$ are not functions of c_s, we have

$$\frac{\partial U}{\partial c_s} = -\frac{(h+\gamma)}{\left(h+\gamma+\widetilde{\lambda}\sigma(c_s)\right)^2}\left(h+\gamma+(c_i-c_s)\widetilde{\lambda}\frac{\partial \sigma}{\partial c_s}+\widetilde{\lambda}\sigma(c_s)\right)$$

From calculus, we know that an interior best response must occur when the derivative of the utility is zero. Therefore, by solving $\frac{\partial U}{\partial c_s} = 0$ for c_s^B and rearranging, we arrive at the geometric condition

$$(c_s^B - c_i)\frac{\partial \sigma}{\partial c_s} = \frac{h+\gamma}{\widetilde{\lambda}} + \sigma(c_s^B, c_t). \tag{6}$$

The right hand side of Eq. (6) is always positive, so equality requires $c_s^B \in [0, c_i)$. Furthermore, any line relating c_s to σ through the point $(c_i, -(h+\gamma)/\widetilde{\lambda})$ is a solution. For a fixed public investment rate c_t, if the best response $c_s^B > 0$, then the line drawn between $(c_s^B, \sigma(c_s^B, c_t))$ and $(c_i, -(h+\gamma)/\widetilde{\lambda})$ must be tangent to the curve $\sigma(c_s, c_t)$ at $c_s = c_s^B$ on the plane of c_s versus σ. Depending on the shape of σ (see Fig. 1), there may be several points satisfying this necessary condition, in addition to the boundary point $c_s = 0$. If the cost of small enough, the best response will be to do nothing ($c_s^B = 0$). If the relative exposure rate is a convex function of the individual investment, then we can see geometrically that there is always a unique best response (see the left sub-plot in Fig. 1). Moreover, the geometry can be summarized as follows. (See Appendix A for the proof)

Theorem 1. *If the relative exposure rate $\sigma(c_s, c_t)$ is convex and decreasing in individual investment c_s, then there is always a unique best response, and this best response increases with both the cost of disease and the infection pressure.*

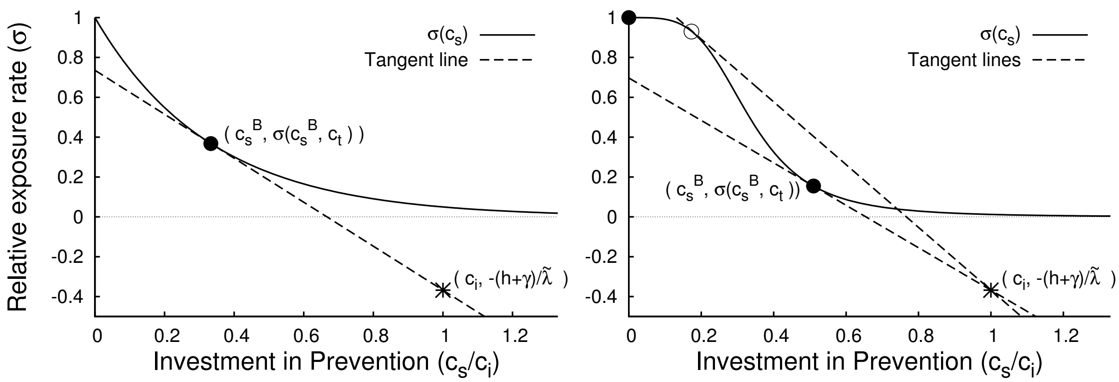

Figure 1: These plots show two tangent-line constructions of points satisfying the necessary differential condition for a best response. First, we plot the relative exposure rate $\sigma(c_s, c_t)$ as a function of the individual's investment c_s for a fixed public investment c_t. Then we draw all tangent lines to $\sigma(c_s, c_t)$ through the point $(c_i, -(h+\gamma)/\widetilde{\lambda})$. The points of tangency satisfy the necessary condition. In general, there may be multiple points of tangency (right), corresponding to multiple local extremes, but strict convexity (left) is enough to guarantee uniqueness. If the initial slope is not too negative, $c_s = 0$ may also satisfy the necessary condition for a best response (right). At the open dot, the necessary condition is satisfied, but is a local minimum rather than a maximum. [TCR: consider adding $U(c_s)$ subplot on top, for reference -- need to recreate utility function, though, in python code]

Our study of population games has show that it will be convenient to think of the best response as a *correspondence* depending on the typical investment $\overline{c}_s$ and public investment c_t: $c_s^B(\overline{c}_s, c_t) := \text{argmax}_{c_s \geq 0} U(c_s, \overline{c}_s; c_t)$. A correspondence is a set-valued map whose output may contain 0,1,2, or more values. So, depending on the typical investment and the public investment, there is a set of best responses an individual can choose from. The typical investment $\overline{c}_s$ appears in Eq. (6) only implicitly through the stationary infection pressure $\widetilde{\lambda}$, while the policy investment c_t appears in both the relative exposure rate σ and $\widetilde{\lambda}$. The stationary infection pressure $\widetilde{\lambda}$ depends on the typical investment $\overline{c}_s$, but not the individual investment c_s. To emphasis this dependence pattern, we will henceforth denote the best-response correspondence as $c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t))$.

3.2 Game Equilibria

We would now like to use the best-response correspondence to identify equilibria for individual investment. A strategy c_s^* is a game equilibrium if it is a best response to itself, i.e., it satisfies the set inclusion relation

$$c_s^B(c_t, \widetilde{\lambda}(c_s^*, c_t)) \ni c_s^*. \tag{7}$$

The best-response correspondence may be discontinuous and undefined for specific parameter values, so in general, we can not be sure that a solution c_s^* to Eq. (7) exists. But under the assumption that $\sigma(c_s, c_t)$ is strictly convex in c_s, our tangent-line construction in Fig. 1 shows that the best response exists and is unique. Since there is then exactly one best response, we can convert Eq. (7) to

$$c_s^B(c_t, \widetilde{\lambda}(c_s^*, c_t)) = c_s^*. \tag{8}$$

There is a unique game equilibrium behavior under the assumption that σ is convex with respect to the first argument, i.e., c_s. This can be shown by first establishing monotonicity of the infection pressure with respect to the typical investment and public investment. (See Appendix A for the proofs of Theorems 2 and 3)

Theorem 2. *If $\sigma(\overline{c}_s, c_t)$ is decreasing, then $\widetilde{\lambda}(\overline{c}_s, c_t)$ and $\sigma(\overline{c}_s, c_t)\widetilde{\lambda}(\overline{c}_s, c_t)$ are decreasing or flat in both $\overline{c}_s$ and c_t.*

From this, we can show the following theorem.

Theorem 3. *If $\sigma(\overline{c}_s, c_t)$ is decreasing and convex in $\overline{c}_s$, then there is a unique game equilibrium $c_s^*(c_t)$.*

The easiest way to determine the game equilibria when σ is smooth is to combine Eq. (2), Eq. (6) and Eq. (8) to identify the strategy that is a best response to its own infection pressure:

$$(c_s^* - c_i)\frac{\partial \sigma}{\partial c_s}(c_s^*, c_t) = \frac{(h+\gamma)}{\beta - \gamma/\sigma(c_s^*, c_t)} + \sigma(c_s^*, c_t) \tag{9}$$

for an interior equilibrium. Both the existence-uniqueness and the necessary condition can be directly generalized to piece-wise smooth functions with equivalent convexity-properties.

In Appendix A, we can go further to prove that the equilibrium is an evolutionary stable strategy (Theorem 4). In Appendix B, we construct the equilibrium for a specific case as well as a general bound independent of the geometry of the relative exposure rate $\sigma(\overline{c}_s, c_t)$ (Theorem 6). We also show that in this model, community health is always vulnerable to free-riding [26]. (Theorem 7).

4 Impacts of Public Investment

Game equilibria provide us with a description of individuals' rational response to the risks imposed by an endemic infectious disease, depending on the specific relationship between individual investment and reductions in their relative exposure rate. But the equilibrium is also a function of the impacts of public investment, so we may say $c_s^*(c_t)$. In this section, we will study how the effects of public investment described by $\sigma(c_s, c_t)$ determine $c_s^*(c_t)$ and what the consequences of this relationship are for policy choices.

The effects we are most concerned with are policy resistance and policy reinforcement. Policy resistance and policy reinforcement describe feedbacks between small changes in public investment and the public's response to these changes. Suppose $W(\overline{c}_s; c_t) := U(\overline{c}_s, \overline{c}_s; c_t)$ represent the typical utility to an individual playing the typical strategy under an existing policy c_t and we are considering a small change Δc_t to this policy. Policy reinforcement describes situations where the direct effects of the proposed change improve the typical utility, and feedbacks from changes in the game equilibrium in response to the policy change are also positive:

$$\frac{\partial W}{\partial c_t}\Delta c_t > 0 \quad \text{and} \quad \frac{\partial W}{\partial c_s^*}\frac{\partial c_s^*}{\partial c_t}\Delta c_t > 0 \tag{10a}$$

where

$$\frac{\partial W}{\partial c_s^*} := \left[\frac{\partial U}{\partial c_s} + \frac{\partial U}{\partial \overline{c}_s}\right]_{c_s = c_s^*, \overline{c}_s = c_s^*} = \frac{\partial U(c_s^*, c_s^*; c_t)}{\partial \overline{c}_s}. \tag{10b}$$

(The derivative with respect to c_s vanishes because c_s^* is a best response.) The feedback dependence on c_t appears because changes in public investment can alter the efficiency of individual investment. Policy resistance describes situations where the direct effects of the proposed change improve the typical utility, but the feedbacks from changes in the game equilibrium in response to the policy change are negative:

$$\frac{\partial W}{\partial c_t}\Delta c_t > 0 \quad \text{and} \quad \frac{\partial W}{\partial c_s^*}\frac{\partial c_s^*}{\partial c_t}\Delta c_t < 0. \tag{11}$$

Policy resistance creates situations where a policy might fail despite having positive direct effects because the negative effects from indirect feedbacks outweigh the direct effects:

$$\frac{\partial W}{\partial c_t}\Delta c_t > 0 \quad \text{but} \quad \left(\frac{\partial W}{\partial c_t} + \frac{\partial W}{\partial c_s^*}\frac{\partial c_s^*}{\partial c_t}\right)\Delta c_t < 0. \tag{12}$$

We must emphasize that the concepts of policy-reinforcement and policy resistance are purely from the perspective of the social planner, and do not consider public preference for policy change. Individuals, for example, may strongly support policy changes leading to policy resistance.

Formulas for changes in equilibrium play can be derived using the algebra of infinitessials and can be expressed in terms of changes in the marginal rate of return [7] --

$$\frac{\partial c_s^*}{\partial c_t} = \frac{\dfrac{\partial^2 U}{\partial c_s \partial c_t}}{-\left(\dfrac{\partial^2 U}{\partial c_s^2} + \dfrac{\partial^2 U}{\partial c_s \partial \overline{c}_s}\right)} \tag{13}$$

The denominator is positive whenever the game equilibrium is convergently stable [12], which Theorem 4 ensures, so the sign of change is determined by the sign of the numerator. If $\partial^2 U/\partial c_s \partial c_t < 0$, we say that public investment is a strategic substitute for individual investment, and more public investment decreases individual investment. On the other hand, if $\partial^2 U/\partial c_s \partial c_t > 0$, public investment is a strategic complement of individual investment, and public investment facilitates individual investment. Since $c_s^* < \overline{c}_s^*$, Theorem 7 implies $\frac{\partial W}{\partial c_s^*} > 0$. Thus, we can conclude that policy resistance arises when public and individual investment are strategic substitutes at equilibrium, while policy reinforcement arises when public and individual investment are strategic complements. However, to make this nomenclature useful, we must delve deeper into the mechanistic details underlying management practices.

If disease transmission is grossly under control, and the efficiency of individual investment diminishes as public investment increases, then public investment will cause policy-resistance. (see Appendix C for the proof of Theorem 8)

Theorem 8. *Assume the relative exposure rate function* $\sigma(c_s, c_t)$ *satisfies the following conditions:*

(H1) σ *is decreasing in* c_s *and* c_t*, smooth and convex with respect to* c_s*, and* $\frac{\partial^2 \sigma}{\partial c_t \partial c_s} > 0$*;*

(H2) $1 \leq \sigma \mathcal{R}_0 \leq 2$*.*

Then increased public investment decreases equilibrium individual investment in self-protection ($\mathrm{d}c_s^*/\mathrm{d}c_t \leq 0$)*.*

The combination of Theorem 8 and Theorem 7 implies policy resistance (see Appendix C Corollary 1) However, policy resistance is not universal -- if the exposure rate is large ($\sigma(\bar{c}_s, c_t)\mathcal{R}_0 > 2$) or if increased public investment facilitates individual investment by increasing its efficiency, public investment may lead to policy reinforcement. In fact, the geometry of σ strongly influences the potential for policy resistance or policy reinforcement responses. To explore the possibilities, we will look at three different classes of relations between investment and the relative exposure rate, based on possible public health interventions and accounting for the interplay between government and individual actions.

4.1 Independent Interventions

One of the simplest possibilities for the relationships between individual and government actions would be that preventing the disease transmission requires a series of factors and that individual and government interventions can influence non-overlapping subsets of these factors. Then individual and public investments independently reduce the exposure rate. This implies that the relative exposure rate can be decomposed into a product, $\sigma(c_s, c_t) = \sigma_s(c_s)\sigma_t(c_t)$. This relatively simple assumption implies that Theorem 8 can be extended globally -- the equilibrium investment rate will always be reduced in response to increased public investment and small policy improvements will always face policy resistance. (see Appendix C for the proof of Theorem 9)

Theorem 9. *If the effects of government and individual interventions are independent, such that* $\sigma(c_s, c_t) = \sigma_s(c_s)\sigma_t(c_t)$*, and* $\sigma_s(c_s)$ *is smoothly decreasing and convex, then increased public investment decreases equilibrium individual investment in self-protection* ($\mathrm{d}c_s^*/\mathrm{d}c_t \leq 0$)*.*

Combined with Theorem 7, Theorem 9 implies independent public-health interventions always face policy resistance, i.e., the returns on public investment will always be diminished by feedbacks from the public response. Whether or not policy-failure occurs depends on the specifics of σ. As Fig. 2 illustrates, sometimes the best public investment may be so large that all individual investments have stopped.

While it is frequently impossible to obtain closed-form expressions for the equilibria of our model, there are exceptions, particularly in the case of independent interventions. For example, if

$$\sigma(c_s, c_t) = \begin{cases} \sigma_t(c_t)\,(1 - c_s/c_i)^n & \text{if } 0 \le c_s < c_i,\, n > 1, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

then Eq. (9) can be solved exactly. We do not have a mechanistic motivation for Eq. (14) but it is consistent with our economic hypotheses that the relative exposure rate be decreasing and convex as long as $\sigma_t(c_t)$ is decreasing and convex. The solution provides us with the following unique equilibrium individual investment rate, as predicted by Theorem 3,

$$c_s^*(c_t) = c_i \max\left\{0, 1 - \left[\frac{h + n\gamma}{(n-1)\beta\sigma_t(c_t)}\right]^{1/n}\right\}. \tag{15}$$

As expected, the equilibrium investment rate decreases as the government investment rate increases. Also, faster discounting of future returns reduces the equilibrium investment, while faster transmission increases the equilibrium investment rate.

4.2 Facilitative Interventions

On the other hand, public investment may facilitate new opportunities for individuals. Facilitation is defined by a local condition that small increases in public investment make individual investment more efficient: in the sense that the rate of change of the relative exposure rate with respect to the individual investment is negative and decreases more quickly as public investment increases ($\frac{\partial \sigma}{\partial c_s} < 0$, $\frac{\partial^2 \sigma}{\partial c_t \partial c_s} < 0$). This violates hypothesis H1 of Theorem 8. Independent interventions are never facilitative, but there are many other conditions where the relative exposure rate allows facilitation in response to interventions. Facilitative interventions can exhibit policy reinforcement, where increased public investment promotes greater individual investment (See Fig. 3). For example, public investment may not affect exposure rates directly, but could provide new opportunities accessible to individuals. This is commonly the case for public investment in education: the investment does not directly affect susceptibility, but helps individuals to adopt more efficient strategies of reducing relative exposure rates(e.g., best practices for sanitation, hygiene, social distancing). Another case may be government interventions that provide alternative sources of drinking water, but individuals have the freedom to choose which source to use.

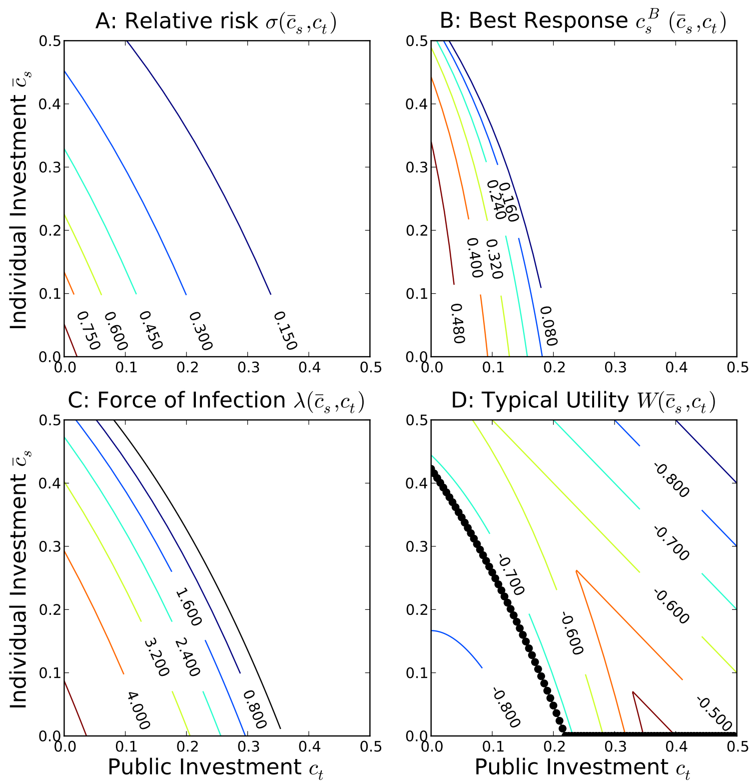

Figure 2: Contour plots of (A) the relative exposure rate $\sigma(\bar{c}_s, c_t)$, (B) the best responses $c_s^B(\bar{c}_s, c_t)$, (C) the infection pressure $\widetilde{\lambda}(\bar{c}_s, c_t)$, and (D) the typical utility $W(\bar{c}_s; c_t)$ when $\mathcal{R}_0 = 6$, $c_i = 1$, and the relative exposure rate function depends on independent interventions by individuals and government and is specified as $\sigma := e^{-5c_t}(1 - c_s)^2$ while utility gains $u(j - c_t) := -c_t$. The dots in (D) represent the equilibria response to policy. This illustrates policy resistance; the equilibrium response decreases as public investment increases. In this particular case, the best policy is eradication exclusively through public investment.

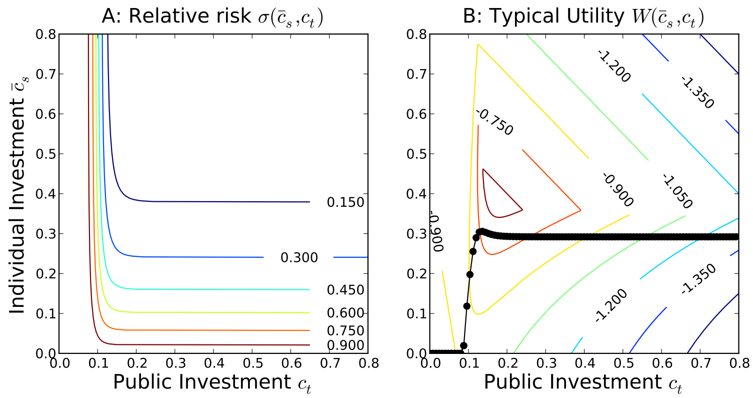

Figure 3: An example where government investment has facilitative effects on relative exposure rate. Contour plots for (A) the relative exposure rate $\sigma(\bar{c}_s, c_t) := (1 - e^{-5c_s})/[1 + (10c_t)^8] + e^{-5c_s}$ and (B) the typical utility $W(\bar{c}_s; c_t)$ on the plane of $\bar{c}_s$ versus c_t when $\mathcal{R}_0 = 6$, $c_i = 1$, and $u(j - c_t) = -c_t$. The dots in (B) represent the game equilibria. The shape of σ ensures the uniqueness of game equilibrium. However, different from the previous example, this example illustrates policy reinforcement, where increasing public investment can promote individual investment.

4.3 Exchangeable Interventions

A third relationship is one where individual and government interventions act on the same factors in equivalent ways; a decrease in investment by one can be exactly offset by a proportional increase in investment by the other. Interventions are exchangeable when $\sigma(c_s, c_t) = \psi(a_s c_s + a_t c_t)$ for some convex decreasing function $\psi(\cdot)$ and positive constants a_s and a_t. We might expect exchangeable investments to act as perfect strategic substitutes. Indeed, we can show that Theorem 8 applies. However, this does not extend to a global result as in the case of independent interventions. Despite the fact that public and individual investments are exchangeable within the management machinery, there are cases where equilibrium investments in self-protection will be increased in response to increased public investment ($\mathrm{d}c_s^*/\mathrm{d}c_t > 0$). Combined with Theorem 7, this implies strategic complementarity and policy reinforcement in response to public investment -- sometimes it is better to have the government help, even if one can do it for oneself. Such an example is shown in Figs. 4 and 5 when $\sigma(c_s, c_t) = 1 - 0.85\,[1 + (1.2c_s + 1.2c_t)^{-6}]^{-1/6}$. This somewhat unusual functional form is a mollification of a piecewise linear function, and it is an open question if such forms are more than mathematical curiosities in this context.

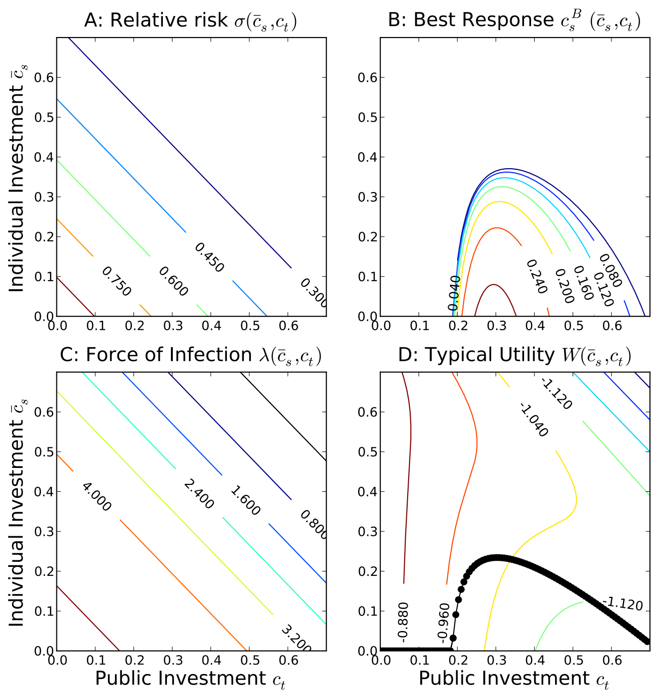

Figure 4: An example where both policy resistance or policy reinforcement occur under exchangeable investments. Here, σ is given by $\sigma(c_s, c_t) := 1 - 0.85\left[1 + (1.2c_s + 1.2c_t)^{-6}\right]^{-1/6}$ while utility gains $u(j - c_t) = -c_t$. Plots A, B, C, D are the contour plots on the plane of $\bar{c}_s$ versus c_t for the relative exposure rate function σ, best responses $c_s^B(\bar{c}_s, c_t)$, the infection pressure $\widetilde{\lambda}(\bar{c}_s, c_t)$, and the typical utility $W(\bar{c}_s, c_t)$ when $\mathcal{R}_0 = 6$, $c_i = 1$. As what we see in Plot D, the dots represent the game equilibria, where we observe the shape of σ ensures the uniqueness of game equilibrium. Moreover, for small c_t where σ is big, we observe the policy reinforcement; however, for big c_t where σ is small, we instead observe the policy resistance.

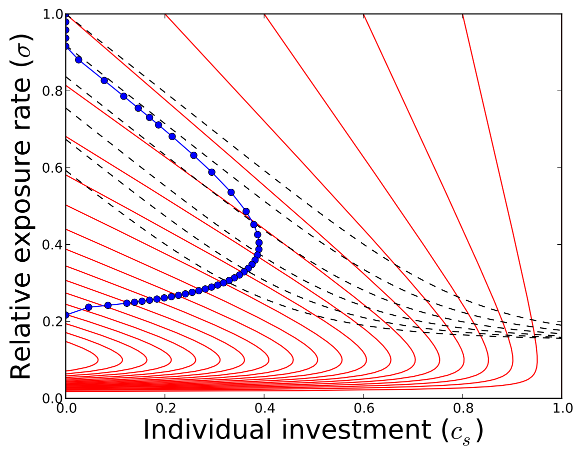

Figure 5: Similar to Fig. 6, this is a simultaneous plot for the relative exposure rate as a function of individual investment for the given σ, and the orbits (solid lines) of Eq. (9) used to identify game equilibria. The dashed lines represent the given function σ with different values c_t. The graph of the given σ moves leftwards as c_t increases. The dots are the game equilibria corresponding to tangent points between these two families of curves for various values of c_t. As the diagram illustrates, when c_t increases, the graph of the given σ moves leftwards, and the game equilibrium first moves rightwards from 0. This implies that $\overline{c}_s^*$ increases as c_t increases, where we observe the policy reinforcement in the sense that increasing the public investment will actually increase the equilibrium individual investment in self-protection. When c_t further increases, $\overline{c}_s^*$ will instead move leftwards approaching 0. This implies that $\overline{c}_s^*$ decreases as c_t increases, and we observe the policy resistance.

5 Discussion

Models can help public health planners improve infectious disease management [1, 13]. Our work here is an attempt to move these theories forward another step by presenting a complete calculation of how the outcomes of public and private investment patterns depend on the geometric properties of interventions and the underlying epidemiological system dynamics. The coordination between public and private actions fundamentally shapes their impact on community health.

We have presented a game played by individuals in a closed population under pressure from an infectious disease with an SIS transmission cycle. This game captures the interplay between individual and public investment in community health through their influence on the relative exposure rate. Based on our construction of equilibrium strategies in terms of relative exposure rates, we can mathematically define the concepts of policy reinforcement and policy resistance. Our analysis shows independent actions lead to policy resistance, though not necessarily policy failure, while facilitative and even exchangeable interventions can create policy reinforcement.

The concept of management in the context of a health commons, as we have explored it, straddles uncomfortable ground across fields of economics, medicine, and population biology. Economically, the ‘‘health commons’’ describes shared states that do not fit the modern formalism of public or private resources; health is not rivalrous, since nobody wants to be sick, and it does not seem to make sense to classify health as excludable or non-excludable. Yet, it is clear, particularly with infectious diseases, that in the very ways William Lloyd described in 1832 [21], illness is something we unwillingly share with those around us, as external costs to our actions which we are selves do not bear. Regardless of our choice of nomenclature, we have shown that it is feasible to make a full accounting of the situation, atleast in theory.

Popular dialog presents an overly simplistic spectrum of the value of public investment. A good understanding of the value of public investment and government intervention requires at least a basic understanding of system dynamics and incentives. This understanding can be used to gain new insights into trends in cases of childhood diarrhea in various locations like Haiti and Thailand (Marisa Eisenberg, personal communication). Public and private actors can engage in a variety of actions aiming to reduce disease transmission. The simultaneous alignment of these public and private actions can generate policy failure or reinforcement. Further, locating the critical interactions is not self-evident or intuitive; systematic models can provide a means by which to locate them. While our policy outcome calculations were performed in a rather elementary example, we believe that a menagerie of similar geometric rules exists for many more practical policy questions related to public health and infectious disease management such as reducing childhood diarrhea. To obtain these results, we will need an engineered approach to policy design, with the patience to account for the mechanisms and feedbacks within a system, be they biological, economic, or behavioral. Sound models combining empirical evidence with systems analyses will help free us from over-simplistic paradigms and provide clearer pictures of the limitations and the opportunities in policy selection. In particular, this approach may help resolve the potential role of health in poverty-traps and economic mobility [4].

The theory we have presented here is an equilibrium theory -- the infectious disease is assumed to be at steady-state prevalence, strategies are static, there are no demographic effects, and government balances its expenditures through taxation rather than borrowing. It has many shortcomings. If the epidemiological state is not stationary, both policy makers and individuals should adopt strategies that are time-dependent. Differential game theory and inductive game theory provide options for analysis of these situations. While stationary average states may suffice for many applied policy-design problems, we should always be concerned about the possibilities of resonance effects, instabilities, and un-anticipated structural issues emerging from complex systems. Even with these limitations, this paper provides important insights into how we can think about public health management. Efforts to address public health problems, like childhood diarrhea [33], can focus not only on what to do and how to scale it up [34], but when to do it. Put differently, the impact of what (e.g., changing sanitation or hygiene) we do to impact transmission may have much to do with when and how it is done.

Acknowledgments

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A Existence, Uniqueness, and Evolutionary Stability

The mathematical analysis in the paper revolves around the analysis of the utility function, which is the sum over all future times of the probability of being in each state times the value of that state. The values of that states in the future are discounted so that they are worth less than the present values. Mathematically,

$$U(c_s, \overline{c}_s; c_t) := \int_0^\infty e^{-ht}\mathbf{v}\cdot\mathbf{p}(t)\mathrm{d}t = \int_0^\infty e^{-ht}\mathbf{v}e^{\widetilde{\mathbf{Q}}t}\mathbf{p}_0\mathrm{d}t$$

Here, we have made use of our model and assumptions to replace $\mathbf{p}(t)$ with its matrix-exponential representation. The discount rate h describes how much less future-returns are worth, compared to present returns. In economics, h may be an interest rate or inflation rate. In evolutionary biology, h is the exponential rate of population growth. Performing the needed integration, we determine that Eq.(5) is given as

$$\begin{aligned}
&U(c_s, \overline{c}_s; c_t) \\
&\quad = \mathbf{v}\left(h\mathbf{I} - \widetilde{\mathbf{Q}}\right)^{-1}\mathbf{p}_0 \\
&\quad = \left[u(j-c_t) - c_s, u(j-c_t) - c_i\right]\left(h\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-\sigma(c_s,c_t)\widetilde{\lambda} & \gamma\\ \sigma(c_s,c_t)\widetilde{\lambda} & -\gamma\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\0\end{bmatrix} \\
&\quad = \left[u(j-c_t) - c_s, u(j-c_t) - c_i\right]\left(\frac{1}{h(h+\gamma+\widetilde{\lambda}\sigma(c_s,c_t))}\begin{bmatrix}h+\gamma & \gamma\\ \sigma(c_s,c_t)\widetilde{\lambda} & h+\widetilde{\lambda}\sigma\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix} \\
&\quad = \frac{u(j-c_t)}{h} - \frac{(h+\gamma)c_s + \widetilde{\lambda}(\overline{c}_s,c_t)\sigma\left(c_s,c_t\right)c_i}{h\left[h+\gamma+\widetilde{\lambda}(\overline{c}_s,c_t)\sigma\left(c_s,c_t\right)\right]}
\end{aligned}$$

Theorem 1. *If the relative exposure rate is smooth, convex, and decreasing in individual investment, then there is always a unique best response for individuals, and this best response increases with both the cost of disease and the infection pressure.*

Proof. First, we repeat our argument for the existence of a unique best response when the relative exposure rate is convex and decreasing.

An individual's best response c_s^B maximizes the utility under given the typical investment $\overline{c}_s$ and public investment c_t is the $c_s^B(\overline{c}_s, c_t) := \operatorname{argmax}_{c_s \geq 0} U(c_s, \overline{c}_s; c_t)$. The best response is chosen so that the marginal cost of preventive investment equals the marginal benefit of less frequent infection. Differentiation of U by c_s leads to the geometric condition

$$(c_s^B - c_i)\frac{\partial\sigma}{\partial c_s} = \frac{h+\gamma}{\widetilde{\lambda}} + \sigma(c_s^B, c_t). \tag{16}$$

Note that $\widetilde{\lambda}$ is fixed -- it depends on the typical investment $\bar{c}_s$, not the individual investment c_s. Since $\bar{c}_s$ appears in Eq. (16) only implicitly through $\widetilde{\lambda}$, while c_t appears in σ and $\widetilde{\lambda}$, we will represent the best response's dependencies by $c_s^B(c_t, \widetilde{\lambda}(\bar{c}_s, c_t))$.

The right hand side of Eq. (16) is always positive, so equality requires $c_s^B \in [0, c_i)$. Furthermore, any line relating c_s to σ through the point $(c_i, -(h+\gamma)/\widetilde{\lambda})$ is a solution. For a fixed public investment rate c_t, if the best response $c_s^B > 0$, then the line drawn between $(c_s^B, \sigma(c_s^B, c_t))$ and $(c_i, -(h+\gamma)/\widetilde{\lambda})$ must be tangent to the curve $\sigma(c_s, c_t)$ at $c_s = c_s^B$ on the plane of c_s versus σ. Depending on the shape of σ (see Fig. 1), there may be several points satisfying this necessary condition, possibly including the boundary point $c_s = 0$. If the relative exposure rate is a convex function of the individual investment, then we can see geometrically that there is always a unique best response (see the left sub-plot in Fig. 1). If infection cost is small enough, then no tangent line satisfying our criteria will exist, and $c_s^B = 0$ will be the unique best response.

The observations that the best response increases with the cost of disease and the risk of infection can be shown with calculus when σ is smooth. Re-arranging (16) and differentiating with respect to c_i, we obtain

$$-\frac{\partial \sigma(c_s^B, c_t)}{\partial c_s} + (c_s^B - c_i)\frac{\partial^2 \sigma(c_s^B, c_t)}{\partial c_s^2}\frac{\partial c_s^B}{\partial c_i} = 0. \tag{17}$$

By the monotonicity and convexity of the function $\sigma(c_s, c_t)$ in c_s, we know that

$$\frac{\partial \sigma(c_s^B, c_t)}{\partial c_s} < 0 \quad \text{and} \quad \frac{\partial^2 \sigma(c_s^B, c_t)}{\partial c_s^2} \geq 0. \tag{18}$$

These imply that $\frac{\partial c_s^B}{\partial c_i} > 0$. Similarly, differentiating Eq. (16) with respect to $\widetilde{\lambda}$ and rearranging,

$$(c_s^B - c_i)\frac{\partial^2 \sigma(c_s^B, c_t)}{\partial c_s^2}\frac{\partial c_s^B}{\partial \widetilde{\lambda}} = -\frac{h+\gamma}{\widetilde{\lambda}^2}. \tag{19}$$

By checking the signs of the both sides in the above equation, we know that $\frac{\partial c_s^B}{\partial \widetilde{\lambda}} > 0$. □

Theorem 2. *If $\sigma(\bar{c}_s, c_t)$ is decreasing in both $\bar{c}_s$ and c_t, then $\widetilde{\lambda}(\bar{c}_s, c_t)$ and $\sigma(\bar{c}_s, c_t)\widetilde{\lambda}(\bar{c}_s, c_t)$ are decreasing or flat in both $\bar{c}_s$ and c_t.*

Proof. Suppose we fix $\bar{c}_s$. Then $\sigma(\bar{c}_s, c_t)$ is decreasing in c_t, so $\gamma/\sigma(\bar{c}_s, c_t)$ is increasing and $\widetilde{\lambda} = \max\{0, \beta - \gamma/\sigma(\bar{c}_s, c_t)\}$ must be decreasing in c_t. Since both $\widetilde{\lambda}$ and $\sigma(\bar{c}_s, c_t)$ are non-negative and decreasing or flat in c_t, the product $\sigma(\bar{c}_s, c_t)\widetilde{\lambda}$ must also be decreasing or flat in c_t for fixed $\bar{c}_s$. The argument for $\bar{c}_s$ is the same. □

Theorem 3. *If $\sigma(\overline{c}_s, c_t)$ is decreasing and convex in $\overline{c}_s$, then there is a unique game equilibrium $c_s^*(c_t)$ for every public investment rate $c_t \geq 0$.*

Proof. First, Theorem 1 states that increasing the infection pressure increases the best response ($\partial c_s^B/\partial\widetilde{\lambda} \geq 0$). From Theorem 2, increasing the population's investment $\overline{c}_s$ decreases the infection pressure, so if $c_s^B(c_t, \widetilde{\lambda}(0, c_t)) = 0$, then $c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t)) = 0$ for all $\overline{c}_s > 0$. So $c_s^* = 0$ must be the only strategy that is a best response to itself, i.e., the game equilibrium, and therefore is unique.

On the other hand, suppose $c_s^B(c_t, \widetilde{\lambda}(0, c_t)) > 0$. We observe that $\widetilde{\lambda} < \beta - \gamma$ for all $\overline{c}_s$, implying

$$c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t)) \leq c_s^B(c_t, \beta - \gamma) < \infty. \tag{20}$$

We know that the latter inequality holds because we have shown that $c_s^B \in [0, c_i)$ and the cost of infection c_i is generally finite. Then we have,

$$\lim_{\overline{c}_s \to \infty} \left[c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t)) - \overline{c}_s \right] < 0 < c_s^B(c_t, \widetilde{\lambda}(0, c_t)) - 0. \tag{21}$$

Since $c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t))$ is continuous in $\overline{c}_s$, by the intermediate value theorem of continuous functions, there must be at least one solution to

$$c_s^B(c_t, \widetilde{\lambda}(c_s^*, c_t)) = c_s^*. \tag{22}$$

Since $\partial c_s^B/\partial\widetilde{\lambda} \geq 0$ and $\partial\widetilde{\lambda}/\partial\overline{c}_s \leq 0$, $c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t))$ must be decreasing in $\overline{c}_s$. By the monotonicity of $c_s^B(c_t, \widetilde{\lambda}(\overline{c}_s, c_t))$ with respect to $\overline{c}_s$, there can be no more than one solution to Eq. (22). Thus, there is a unique game equilibrium for $c_s^B(c_t, \widetilde{\lambda}(0, c_t)) > 0$.

We conclude that there is always a unique global game equilibrium for individual behavior under the given assumptions. □

Theorem 4. *If $\sigma(\overline{c}_s, c_t)$ is decreasing and convex in $\overline{c}_s$ for $c_t \geq 0$, then the equilibrium strategy always has invasion potential, and hence is an evolutionarily stable strategy.*

Proof. The argument for invasion potential is less straight forward than that for the Nash condition. Since our argument is independent of c_t, we will simplify our notation by omitting it henceforth. We begin working from our known information. First, since $\lambda(c_s)$ is a non-negative decreasing function,

$$\frac{\overline{c}_s - c_s^*}{\lambda(\overline{c}_s)} \geq \frac{\overline{c}_s - c_s^*}{\lambda(c_s^*)}. \tag{23}$$

Now, since c_s^* is a Nash equilibrium (Theorem 3), we know $U(c_s, c_s^*) - U(c_s^*, c_s^*) \leq 0$. This implies, after a fair bit of algebra, that

$$\left(\frac{c_i - c_s}{\sigma\,(c_s)} - \frac{c_i - c_s^*}{\sigma\,(c_s^*)}\right) + \frac{(h+\gamma)(c_s^* - c_s)}{\sigma\,(c_s)\,\sigma\,(c_s^*)\,\widetilde{\lambda}(c_s^*)} \leq 0. \tag{24}$$

With just a change of sign,

$$\left(\frac{c_i - c_s^*}{\sigma\,(c_s^*)} - \frac{c_i - c_s}{\sigma\,(c_s)}\right) + \frac{(h+\gamma)(c_s - c_s^*)}{\sigma\,(c_s^*)\,\sigma\,(c_s)\,\widetilde{\lambda}(c_s^*)} \geq 0. \tag{25}$$

Now, an equilibrium is an evolutionarily stable strategy (ESS) if it has global invasion potential, in the sense that it improves on any alternative typical behavior. Mathematically, $U(c_s^*, \bar{c}_s) - U(\bar{c}_s, \bar{c}_s) \geq 0$. Well,

$$U(c_s^*, \bar{c}_s) - U(\bar{c}_s, \bar{c}_s) = -\frac{(h+\gamma)c_s^* + \widetilde{\lambda}(\bar{c}_s)\sigma\,(c_s^*)\,c_i}{h\left[h+\gamma+\widetilde{\lambda}(\bar{c}_s)\sigma\,(c_s^*)\right]} + \frac{(h+\gamma)\bar{c}_s + \widetilde{\lambda}(\bar{c}_s)\sigma\,(\bar{c}_s)\,c_i}{h\left[h+\gamma+\widetilde{\lambda}(\bar{c}_s)\sigma\,(\bar{c}_s)\right]} \tag{26}$$

From this, we can show that c_s^* invades universally as long as

$$\left(\frac{c_i - c_s^*}{\sigma\,(c_s^*)} - \frac{c_i - \bar{c}_s}{\sigma\,(\bar{c}_s)}\right) + \frac{(h+\gamma)(\bar{c}_s - c_s^*)}{\sigma\,(\bar{c}_s)\,\sigma\,(c_s^*)\,\widetilde{\lambda}(\bar{c}_s)} \geq 0. \tag{27}$$

After we substitute $\bar{c}_s$ for c_s, Eq. (25) differs from Eq. (27) only in the infection-pressure term. A substitution using Eq. (23) shows us that Eq. (25) implies Eq. (27). So if c_s^* is a Nash equilibrium, it also has invasion potential. Since the strategy satisfies both the Nash condition and the invasion condition, it is an evolutionarily stable strategy. □

B Equilibrium calculation and bounds

It is often impossible to identify a closed-form representation of the game equilibrium c_s^* from Eq. (9). For these cases, one can still identify the game equilibrium using either numerical or geometric approaches when the relative exposure rate function σ is given. First, we can numerically locate the strategy that is a best response to itself directly using the formula for the expected utility, Eq. (5). This is sure to return the unique game equilibrium for individual behavior if function σ satisfies the conditions in Theorem 3. Alternatively, the equilibrium can be located using a phase-plane approach as follows. Eq. (9) can be read as a first-order differential equation for σ in terms of c_s^*, with implicit solutions

$$C = \frac{(\sigma\beta)^{\frac{\gamma}{h}}(c_i - c_s^*)}{(\sigma\beta + h)^{1+\frac{\gamma}{h}}} \text{ if } h > 0 \text{ or } \frac{(c_i - c_s^*)}{\sigma\beta e^{\gamma/(\sigma\beta)}} \text{ if } h = 0. \tag{28}$$

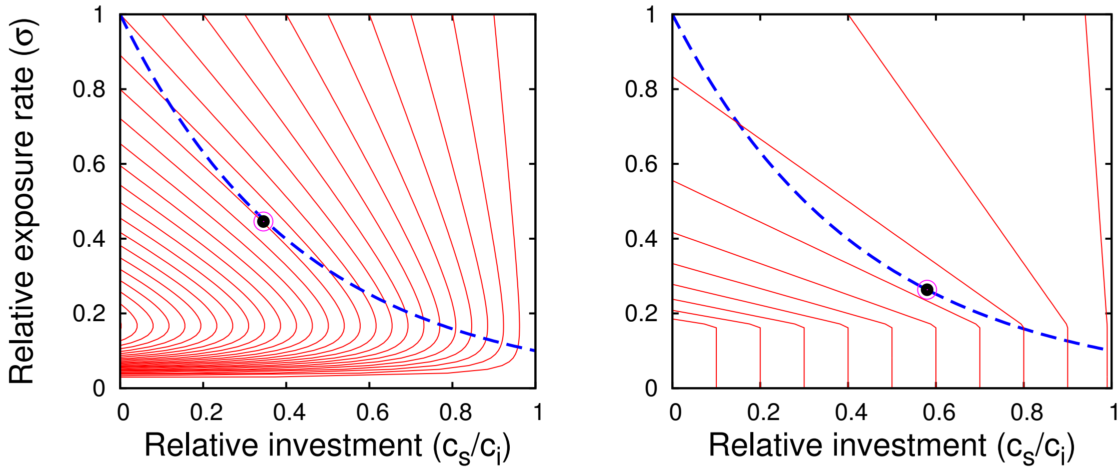

Figure 6: Points where the orbits of Eq. (9) are tangent to the relative exposure rate σ. The dashed line represents the given function σ, while the solid lines represent a family of the orbits to Eq. (9) (left) or contours of constant utility (right). The dot is the Nash equilibrium point c_s^* where σ and the solutions of Eq. (9) are tangent(left) or the socially optimal strategy $\overline{c}_s^*$ (right). This figure shows a special case when $\mathcal{R}_0 = 6$ and $\sigma(c_s, c_t) = \exp(-2.5c_s)$, so $c_s^* \approx 0.34$ and $\overline{c}_s^* \approx 0.58$.

We can now draw level curves representing solutions of the necessary condition for a game equilibrium, Eq. (9). We can also plot $\sigma(c_s, c_t)$ as a function of c_s. By Theorem 3, there must be a point $(c_s^*, \sigma(c_s^*))$ where Eq. (9) holds. Geometrically, this point is the tangent point between $\sigma(c_s, c_t)$ and the curves representing the necessary condition (Fig. 6).

We now see that the nature of the game equilibria for individual behavior depends to some degree on how different investments reduce risk, as specified by the shape of $\sigma(c_s, c_t)$. Still, under the same assumption on the relative exposure rate described in Theorem 3, the game equilibrium will be bounded.

In order to prove the boundedness of the game equilibrium, we will first claim the following theorem, which tells us that for a given relative exposure rate function we can construct a piece-wise linear relative exposure rate function sharing the same game equilibrium.

Lemma 5. *For any given relative exposure rate function $\sigma(c_s, c_t)$ with the properties that σ is decreasing and convex in c_s for fixed c_t, there exists a piece-wise linear function $\sigma_L(c_s) := \max(1 - mc_s, \epsilon)$, $(m \geq 0, \epsilon \geq 0)$ with the same game equilibrium as σ.*

Proof. We will explicitly construct the function of σ_L. For the given function σ, we know that it guarantees the unique existence of game equilibrium c_s^* by Theorem 3. If $c_s^* = 0$, then any σ_L with $\epsilon = 1$ which has any value of c_s including $c_s = 0$ as its equilibria since σ_L is flat in c_s making the value of $c_s = 0$ a best response to any $\bar{c}_s$. If $c_s^* > 0$, we can construct the first piece of σ_L by connecting the points of $(0, 1)$ and $(c_s^*, \sigma(c_s^*))$ in the plane of c_s versus σ for any fixed c_t. The second piece will be the horizontal ray starting at point $(c_s^*, \sigma(c_s^*))$. Therefore, take $-m$ to be the slope of the line between points $(0, 1)$ and $(c_s^*, \sigma(c_s^*))$, and $\epsilon = \sigma(c_s^*)$. We then have a piece-wise linear function $\sigma_L = \max(1 - mc_s, \epsilon)$.

Now we show that σ_L guarantees the same game equilibrium as σ. By the construction of σ_L, we know that σ_L is located in the convex hull of the set $\{(c_s, \sigma(c_s)) : c_s \geq 0\}$ which allows σ_L to have the same tangent property as σ at point $(c_s^*, \sigma(c_s^*))$. This implies that, as described in geometrical approach, σ_L will also be tangent with the same solution curve to

$$\sigma(c_s^*, c_t) + (c_i - c_s^*)\frac{\partial \sigma}{\partial c_s}(c_s^*, c_t) = \frac{-(h+\gamma)}{\beta - \gamma/\sigma(c_s^*, c_t)} \tag{29}$$

as σ at point $(c_s^*, \sigma(c_s^*))$. In other words, c_s^* is also the game equilibrium for σ_L. (Fig. S2) $\square$

Thus, by Lemma 5, if we find the set of possible game equilibria for all relative exposure rate functions of the form of σ_L, then this is also the set of possible game equilibria for all σ. We can use a geometric argument similar to that presented for smooth functions σ (Fig. 6) to find the game equilibria for the piece-wise linear functions $\sigma_L(c_s)$. For convenience, we introduce

$$\hat{m} := -\left.\frac{\partial \sigma}{\partial c_s}\right|_{c_s = c_t = 0, \sigma = 1} = \frac{h+\beta}{c_i(\beta - \gamma)}. \tag{30}$$

as notation for the minimum efficiency below which no internal equilibrium exists, based on Eq. (29).

Now for any piece-wise linear function $\sigma_L = \max(1 - mc_s, \epsilon)$ with $(m \geq 0, \epsilon \geq 0)$, let us consider its possible game equilibria. If $m < \hat{m}$, then there are no points where σ_L is tangent to any of the phase-plane orbits of Eq. (29). As such, $c_s^* = 0$ is the only game equilibrium. If $m > \hat{m}$, we will begin with a ray starting at the point $(0, 1)$ with slope $\hat{m}$, and then locate the tangent point where the ray is tangent with any Nash equilibrium solution curve of Eq. (29). (The existence of the tangent point follows from a property of the solution curves of Eq. (29). The slope of the solution curves decreases to the negative infinity when $\sigma\mathcal{R}_0 = 1$ as c_s increases, as seen in Fig. 6. By the intermediate-value theorem, there is a point where the solution curve has a slope with m.). This point $(\hat{c}_s, \hat{\sigma}(\hat{c}_s))$ is the solution to the system

$$\hat{\sigma} = 1 - m\hat{c}_s, \quad \hat{\sigma} + (c_i - \hat{c}_s)(-m) = \frac{-(h+\gamma)}{\beta - \gamma/\hat{\sigma}}. \tag{31}$$

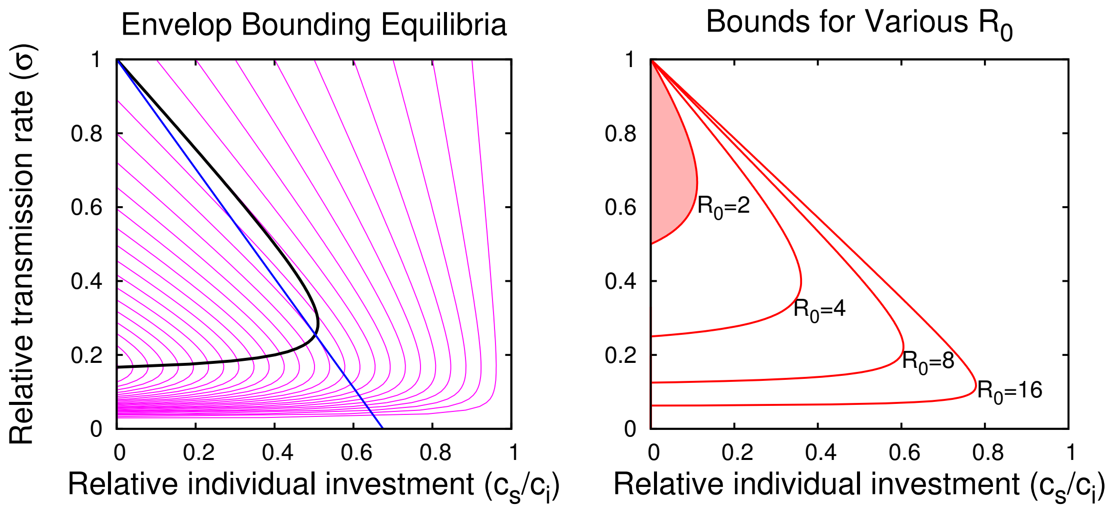

Figure 7: Left: The demonstration to show the determination of the general bound of the game equilibria. Extreme bounds of the set of possible game equilibria for $\mathcal{R}_0 = 6$ (dashed line). Extreme equilibria occur on lines through the point $c_s = 0, \sigma = 1$ (dotted line) are tangent to orbits of the necessary differential condition for game equilibria (solid curves). Right: The plot of the bound on game equilibrium for different values of $\mathcal{R}_0 = \beta/\gamma$ when $h = 0$ (Eq. (32)). Each contour is labeled with the value of $\mathcal{R}_0$ for which it is the bound. Equilibria can exist at each point with the same or smaller value of $\mathcal{R}_0$ for which they are being calculated.

We determine an alternate piece-wise linear function $\hat{\sigma}_L(c_s; m) := \max(1 - mc_s, \hat{\epsilon}(m))$ where $\hat{\epsilon}(m) := \hat{\sigma}(\hat{c}_s)$. If $\epsilon \leq \hat{\epsilon}$, by the construction of $\hat{\sigma}_L$, we know that the game equilibrium will be the point $(\hat{c}_s, \hat{\sigma}(\hat{c}_s))$ since σ_L and $\hat{\sigma}_L$ share this point of tangency to the orbits of Eq. (29). If $\epsilon > \hat{\epsilon}$, the game equilibrium can only possibly correspond to the corner point, $((1-\epsilon)/m, \epsilon)$, since σ_L can only be tangent (in a geometric sense) to one of the orbits at this corner point. Therefore, $c_s^* \in [0, (1-\epsilon)/m]$. So far, we have identified the set of possible game equilibria for piece-wise linear functions in the form of σ_L. This set will be bounded by the curve of $(\hat{c}_s, \hat{\sigma}(\hat{c}_s))$ determined by Eq. (31). (see the left subplot of Fig. 7) Again, since Lemma 5 shows that every game equilibrium is also an equilibrium for some σ_L, this bound also holds for any σ and c_t.

Summarizing the above discussion, we have the following conclusion.

Theorem 6. *The unique equilibrium c_s^* found in Theorem 3 when σ is convex in c_s is always* *bounded in the sense that*

$$0 \leq \frac{c_s^*}{c_i} \leq \frac{\sigma\gamma - \gamma - \beta\sigma^2 + \beta\sigma}{\sigma h + \sigma\gamma - \gamma + \beta\sigma} \leq \frac{(1-\sigma)(\sigma\mathcal{R}_0 - 1)}{\sigma\mathcal{R}_0 - 1 + \sigma}. \tag{32}$$

Proof. The argument initiated by Lemma 5 leads us to Equation (31) as the bounding condition on equilibria. From the first part, $m = (1-\hat{\sigma})/\hat{c}_s$. When we substitute for m in the second part,

$$\begin{aligned}
\hat{\sigma} - (c_i - \hat{c}_s)(1-\hat{\sigma})/\hat{c}_s &= \frac{-(h+\gamma)}{\beta - \gamma/\hat{\sigma}} \\
-\left(\frac{c_i}{\hat{c}_s} - 1\right)(1-\hat{\sigma}) &= -\frac{(h+\gamma)}{\beta - \gamma/\hat{\sigma}} - \hat{\sigma} \\
\left(\frac{c_i}{\hat{c}_s} - 1\right) &= \frac{(h+\gamma)}{(\beta - \gamma/\hat{\sigma})(1-\hat{\sigma})} + \frac{\hat{\sigma}}{(1-\hat{\sigma})} \\
\frac{c_i}{\hat{c}_s} &= \frac{(h+\gamma)}{(\beta - \gamma/\hat{\sigma})(1-\hat{\sigma})} + \frac{\hat{\sigma}}{(1-\hat{\sigma})} + 1 \\
\frac{c_i}{\hat{c}_s} &= \frac{(h+\gamma)}{(\beta - \gamma/\hat{\sigma})(1-\hat{\sigma})} + \frac{1}{(1-\hat{\sigma})} \\
\frac{\hat{c}_s}{c_i} &= \frac{(\beta - \gamma/\hat{\sigma})(1-\hat{\sigma})}{(h+\gamma) + (\beta - \gamma/\hat{\sigma})}.
\end{aligned}$$

Our preceding argument showed that this had to be an upper bound on the game equilibrium for a given $\hat{\sigma}$, so we now know

$$\frac{c_s^*}{c_i} \leq \frac{\sigma\gamma - \gamma - \beta\sigma^2 + \beta\sigma}{\sigma h + \sigma\gamma - \gamma + \beta\sigma}.$$

The right-hand side is decreasing in h, so the special case of $h = 0$ provides a weaker upper bound that applies for all discount rates. When we take $h = 0$ and substitute $\mathcal{R}_0 = \beta/\gamma$, we find the parsimonious upper bound

$$\frac{c_s^*}{c_i} \leq \frac{(1-\sigma)(\sigma\mathcal{R}_0 - 1)}{\sigma\mathcal{R}_0 - 1 + \sigma}. \tag{33}$$

□

C Theorems on free-riding and policy effects

First, we provide a free-riding theorem.

Theorem 7. *Let typical utility $W(\overline{c}_s; c_t) := U(\overline{c}_s, \overline{c}_s; c_t)$. When σ is convex in c_s, the best public investment rate $\overline{c}_s^* = \operatorname{argmax}_{\overline{c}_s} W(\overline{c}_s; c_t)$ is always greater than the game equilibrium investment ($c_s^* \le \overline{c}_s^*$) and for every $\overline{c}_s \in (0, \overline{c}_s^*)$, $\dfrac{\partial W}{\partial \overline{c}_s} > 0$.*

Proof. The utility of the typical investment rate is given by

$$W(\overline{c}_s; c_t) = \begin{cases} \frac{u-c_t}{h} - \frac{c_i[\beta\sigma(\overline{c}_s, c_t) - \gamma] + \overline{c}_s(h+\gamma)}{h[h+\beta\sigma(\overline{c}_s, c_t)]} & \text{if } \sigma(\overline{c}_s, c_t)\mathcal{R}_0 > 1, \\ \frac{u-c_t-\overline{c}_s}{h} & \text{if } \sigma(\overline{c}_s, c_t)\mathcal{R}_0 \le 1. \end{cases} \tag{34}$$

By inspection, W is decreasing in $\overline{c}_s$ if $\sigma\mathcal{R}_0 \le 1$. So the maximum occurs for some $\overline{c}_s$ such that $\sigma\mathcal{R}_0 \ge 1$. Differentiating W with respect to $\overline{c}_s$ when $\sigma\mathcal{R}_0 > 1$, we find

$$\frac{\partial W}{\partial \overline{c}_s} = \frac{-(\gamma + h)}{h[h + \sigma(\overline{c}_s, c_t)\beta]} \left(1 + \frac{(c_i - \overline{c}_s)\beta}{[h + \sigma(\overline{c}_s, c_t)\beta]} \frac{\partial \sigma}{\partial c_s}\right). \tag{35}$$

Since σ is monotone decreasing, $\dfrac{\partial W}{\partial \overline{c}_s}$ can change sign no more than once for $\overline{c}_s \in [0, c_i)$. Using the same geometric approach applied for best responses, the local maximum occurs at points on σ where the tangent lines pass through the point $(c_i, -h/\beta)$. The geometry shows that if $\sigma(c_s, c_t)$ is convex in c_s, then $\overline{c}_s^*$ is always uniquely defined (possibly, $\overline{c}_s^* = 0$). So $\overline{c}_s^*$ is equal to

$$\min\left\{c_s : 1 = \sigma(c_s, c_t)\beta/\gamma,\ \overline{c}_s : 1 = \frac{-\beta(c_i - \overline{c}_s)}{h + \beta\sigma(\overline{c}_s, c_t)} \frac{\partial \sigma(\overline{c}_s, c_t)}{\partial c_s}\right\}. \tag{36}$$

If $0 \le \overline{c}_s < \overline{c}_s^*$, $\partial W/\partial \overline{c}_s > 0$. Since $\gamma > 0$ implies $-h/\beta > -(h+\gamma)/\widetilde{\lambda}$, the geometry also shows $\overline{c}_s^* \ge c_s^B$ for all best responses c_s^B, and in particular, $\overline{c}_s^* \ge c_s^*$. □

Theorem 8. *Assume the relative exposure rate function $\sigma(c_s, c_t)$ satisfies the following conditions:*

(H1) *σ is decreasing in c_s and c_t, smooth and convex with respect to c_s, and $\frac{\partial^2 \sigma}{\partial c_t \partial c_s} > 0$;*

(H2) $\sigma(c_s^*(c_t), c_t) \in \left(\frac{\gamma}{\beta}, \frac{\gamma + \sqrt{\gamma^2 + h\gamma}}{\beta}\right)$.

Then increased taxation and public reinvestment decreases equilibrium individual investment in self-protection ($\mathrm{d}c_s^*/\mathrm{d}c_t \le 0$).

Note that $\sigma(c_s^*(c_t), c_t) \in \left(\frac{\gamma}{\beta}, \frac{\gamma + \sqrt{\gamma^2 + h\gamma}}{\beta}\right)$ implies $1 \le \sigma\mathcal{R}_0 \le 2$, so this theorem is slightly stronger than the version given in the main text.

Proof. Based on Eq. (29), a game equilibrium c_s^* satisfies

$$(c_s^* - c_i)\frac{\partial \sigma}{\partial c_s} = \frac{\sigma(c_s^*, c_t)(h + \beta\sigma(c_s^*, c_t))}{\beta\sigma(c_s^*, c_t) - \gamma}. \tag{37}$$

For convenience, we define the right-hand side as a function

$$\phi(\sigma) := \frac{\sigma(h + \beta\sigma)}{\beta\sigma - \gamma}. \tag{38}$$

Note that this is hyperbola in σ, with two linear asymptotes, one local minimum, and one local maximum. The conditions of (H2) specify that σ is in the range where ϕ is positive and decreasing.

We proceed by differentiating Eq. (37) with respect to c_t. We find

$$\frac{\mathrm{d}c_s^*}{\mathrm{d}c_t}\frac{\partial \sigma}{\partial c_s} + (c_s^* - c_i)\Big(\frac{\partial^2 \sigma}{\partial c_s^2}\frac{\mathrm{d}c_s^*}{\mathrm{d}c_t} + \frac{\partial^2 \sigma}{\partial c_s \partial c_t}\Big) = \frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\Big(\frac{\partial \sigma}{\partial c_s}\frac{\mathrm{d}c_s^*}{\mathrm{d}c_t} + \frac{\partial \sigma}{\partial c_t}\Big), \tag{39}$$

which can be re-arranged to the form

$$\frac{\mathrm{d}c_s^*}{\mathrm{d}c_t} = \frac{(c_i - c_s^*)\frac{\partial^2 \sigma}{\partial c_s \partial c_t} + \frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\frac{\partial \sigma}{\partial c_t}}{(c_s^* - c_i)\frac{\partial^2 \sigma}{\partial c_s^2} + (1 - \frac{\mathrm{d}\phi}{\mathrm{d}\sigma})\frac{\partial \sigma}{\partial c_s}}. \tag{40}$$

Next, we can calculate

$$\frac{\mathrm{d}\phi}{\mathrm{d}\sigma} = \frac{(\sigma\beta)(\beta\sigma - 2\gamma) - h\gamma}{(\beta\sigma - \gamma)^2}. \tag{41}$$

Assumption (H2) implies that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\sigma} < 0. \tag{42}$$

This, together with Assumption (H1) implies that the denominator of the right-hand side of (40) is negative, the numerator is positive, and finally that $\mathrm{d}c_s^*/\mathrm{d}c_t \leq 0$. □

Corollary 1. *If Theorem 7 holds, then a small increase in public investment that increases public good ($\partial W/\partial c_t > 0$) will also suffer from policy resistance:*

$$\frac{\partial W}{\partial c_t}\Delta c_t > 0 \quad \textit{and} \quad \frac{\partial W}{\partial c_s^*}\frac{\partial c_s^*}{\partial c_t}\Delta c_t < 0. \tag{43}$$

Proof. For a small increase in public investment, $\Delta c_t > 0$. By assumption, then,

$$\frac{\partial W}{\partial c_t}\Delta c_t > 0.$$

Now, we also know that for any Nash equilibrium, $c_s^* \in (0, \overline{c}_s^*)$, so by Theorem 5,

$$\frac{\partial W}{\partial c_s^*} > 0.$$

and from Thereorem 6,

$$\frac{\partial c_s^*}{\partial c_t} < 0.$$

The conclusion follows by inspection. □

Theorem 9. *If the effects of government and individual interventions are independent, such that*

$$\sigma(c_s, c_t) = \sigma_s(c_s)\sigma_t(c_t),$$

and $\sigma_s(c_s)$ is smoothly decreasing and convex, then increased public investment decreases equilibrium individual investment in self-protection ($\mathrm{d}c_s^*/\mathrm{d}c_t \leq 0$).

Proof. Since the game equilibrium $c_s^*(c_t)$ solves Eq. (22),

$$\frac{\partial c_s^B}{\partial c_t} + \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial c_t} + \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial \overline{c}_s}\frac{\partial \overline{c}_s}{\partial c_s^*}\frac{\partial c_s^*}{\partial c_t} = \frac{\partial c_s^*}{\partial c_t}, \tag{44}$$

with $\overline{c}_s = c_s^*$. We can rearrange the equation and show

$$\frac{\partial c_s^*}{\partial c_t} = \left(\frac{\partial c_s^B}{\partial c_t} + \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial c_t}\right)\left(1 - \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial \overline{c}_s}\right)^{-1}. \tag{45}$$

The proof proceeds by showing the right-hand-side of Eq. (45) is never positive.

If $\sigma(c_s, c_t) = \sigma_s(c_s)\sigma_t(c_t)$, then Eq. (16) reduces to

$$(c_s - c_i)\frac{\partial \sigma_s}{\partial c_s} - \sigma_s(c_s) = \frac{h + \gamma}{\widetilde{\lambda}(\overline{c}_s, c_t)\sigma_t(c_t)}, \tag{46}$$

and then the best response satisfies

$$(c_s^B - c_i)\frac{\partial \sigma_s}{\partial c_s} - \sigma_s(c_s^B) = \frac{h + \gamma}{\widetilde{\lambda}(\overline{c}_s, c_t)\sigma_t(c_t)}, \tag{47}$$

Differentiating the above equation with respect to c_t and rearranging, we have

$$\Big(\frac{\partial c_s^B}{\partial c_t} + \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial c_t}\Big)\Big((c_s^B - c_i)\frac{\partial^2 \sigma_s}{\partial c_s^2}\Big) = -\frac{(h+\gamma)(\frac{\partial \widetilde{\lambda}}{\partial c_t}\sigma_t(c_t) + \widetilde{\lambda}\frac{\partial \sigma_t}{\partial c_t})}{\big(\widetilde{\lambda}(\overline{c}_s, c_t)\sigma_t(c_t)\big)^2} \tag{48}$$

By inspection, we know that

$$-\frac{(h+\gamma)(\frac{\partial \widetilde{\lambda}}{\partial c_t}\sigma_t(c_t) + \widetilde{\lambda}\frac{\partial \sigma_t}{\partial c_t})}{\big(\widetilde{\lambda}(\overline{c}_s, c_t)\sigma_t(c_t)\big)^2} \geq 0 \tag{49}$$

and

$$(c_s^B - c_i)\frac{\partial^2 \sigma_s}{\partial c_s^2} \leq 0, \tag{50}$$

Hence,

$$\frac{\partial c_s^B}{\partial c_t} + \frac{\partial c_s^B}{\partial \widetilde{\lambda}}\frac{\partial \widetilde{\lambda}}{\partial c_t} \leq 0. \tag{51}$$

From this, $\partial c_s^B/\partial\widetilde{\lambda} > 0$ in Remark 1, and $\partial\widetilde{\lambda}/\partial\overline{c}_s \leq 0$ in Theorem 2, we can see by inspection of Eq. (45) that $\partial c_s^*/\partial c_t \leq 0$. □

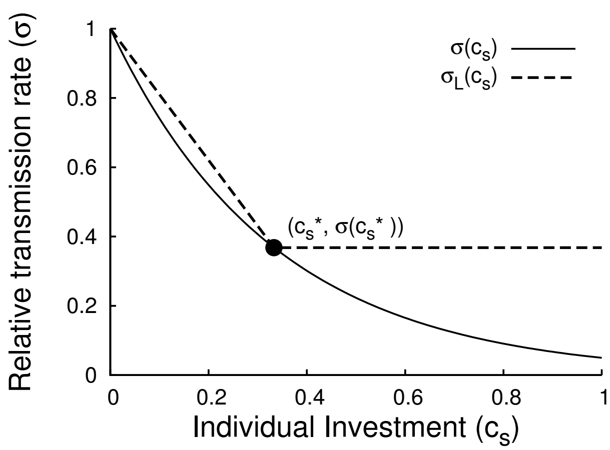

Figure 8: Plot demonstrating the construct of σ_L as used in the proof of Lemma 5. The solid line is the given $\sigma(c_s)$ with fixed c_t and known game equilibrium c_s^*. The dashed line is $\sigma_L(c_s)$. By the construction, c_s^* is always a game equilibrium under σ_L.