# Homework 5

Spring 2025 Math 511, Penn State UP

1. Consider the singularly perturbed second-order differential equation

$$\epsilon y'' = -y' - y$$

with boundary conditions $y(0) = 0$ and $y'(1) = 1$.

   (a) Qualitatively determine the location of the boundary layer.

   (b) Find an approximate boundary-layer solution to the equation $\epsilon y'' = -y' - y$ with boundary conditions $y(0) = -1$ and $y'(1) = 1$.

   (c) Use the phase-space transformation to rewrite this equation as a linear system of first-order equations, and explain the properties of your solution in terms of the phase-plane of this system.

2. Consider the nonlinear delay equation $\dot{n} = rn(t-\tau) - dn^2$ in which recruitment is delayed but density-dependence is instant. Dedimensionalize this equation and then use the boundary-locus method to show dynamics are locally stable around carrying capacity for all $r > 0$, $d > 0$, $\tau > 0$. (Note that this is NOT the same as the Hutchinson-Wright model)

3. The dimensionless predator-prey model of Lotka and Volterra is

$$\dot{n} = rn - np, \quad \dot{p} = np - p.$$

In class, we showed that this system has a neutrally stable steady-state $(1, r)$.

   (a) If recruitment of prey is delayed, how does this alter the stability of co-existence?

   (b) If recruitment of predators is delayed, how does this alter the stability of co-existence?

4. Consider a mass-action chemostat competition system with a single resource,

$$\begin{aligned}
\dot{r} &= \eta - dr - axr - byr \\
\dot{x} &= k_x xr - m_x x \\
\dot{y} &= k_y yr - m_y y
\end{aligned}$$

Derive a quasi-steady-state approximation for this system when resource dynamics are much faster than the population dynamics.

5. Construct an ODE model to explain $\theta(t)$ in the observations of a pendulum over 13 seconds in "https://reluga.org/T/angles.csv".

   (a) Begin with a harmonic oscillator approximation.

   (b) Next try a nonlinear pendulum with damping

   (c) The last remaining piece to explain is an irregularity in the data because of the stand holding up the pendulum string is not perfectly rigid. Use Lagrangian mechanics to derive an improved model that accounts for this, and fit it to the data.